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Generic Singularities of the Optimal Averaged Profit for Polydynamical Systems

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Abstract

We consider the optimization problem that consists in maximizing the time averaged profit for a motion of a smooth polydynamical system on the circle in the presence of a smooth profit density. When the problem depends on a $k$-dimensional parameter the optimal averaged profit is a function of the parameter. We present the classification of all generic singularities of this function when $k \leq 2$.

Resumo

Considera-se o problema de optimiza¸ ç˜ ao que consiste em maximizar o proveito médio temporal para um movimento de um sistema polidinâmico no círculo na presença de uma densidade de proveito. Quando este problema depende de um parâmetro $k$-dimensionnal o proveito médio óptimo é uma função do parâmetro. Apresenta-se a classificação das singularidades genéricas desta função quando $k \leq 2$.

Résumé

Nous considérons le problème d’optimization qui consiste à maximiser le profit moyen temporel pour un mouvement d’un système polydynamique sur le cercle en présence d’une densité de profit. Lorsque ce problème dépend d’un paramètre $k$-dimensionnelle le profit moyen optimal est une fonction du paramètre. Nous présentons la classification de toutes les singularités génériques de cette fonction lorsque $k \leq 2$. 
To my Lovely Parents,
for their inconditional support!

Aos meus Queridos Pais,
pelo seu apoio incondicional!

"Recomeça...
Se puderes,
Sem angústia e sem pressa.
E os passos que deres,
Nesse caminho duro
Do futuro,
Dá-os em liberdade.
Enquanto não alcances
Não descanses.
De nenhum fruto queiras só metade.

E, nunca saciado,
Vai colhendo ilusões sucessivas no pomar.
Sempre a sonhar
E vendo,
Acordado,
O logro da aventura.
És homem, não te esqueças!
Só é tua a loucura
Onde, com lucidez, te reconheças"

Miguel Torga, DIÁRIO XIII
We consider an important optimization problem that was already studied in different ways, namely, the problem of maximizing the time averaged profit for a motion of a smooth control system on the circle in the presence of a smooth profit density. This problem has been brought to Singularity Theory by V.I. Arnold, in 2002, when he introduced a parameter in this problem [5]. In this new approach, the optimal averaged profit is a function of the parameter and it is worthwhile to study its singularities.

On the study of this Singularity Theory problem, A.A. Davydov and H. Mena-Matos proved that there are two types of strategies (motions) that can always provide the optimal averaged profit [12]. These optimal strategies are called stationary strategies and level cycles.

I really have to thank Professor A.A. Davydov for introducing me this problem during a Singularity Theory course and for proposing me to study it in my Master thesis. In this thesis, under the supervision of Davydov and Mena-Matos, it was obtained the classification of all generic singularities of the optimal averaged profit in a simple case, considering polydynamical systems, a 1-dimensional parameter and only one type of optimal strategies, namely, stationary strategies.

After this study I obtained a research grant from Centro de Matemática da Universidade do Porto (CMUP) where it was classified all generic singularities of a very special set included on the study of stationary strategies: the stationary domain. The results are also presented in this work.

Finally, a wider study of this problem was possible by a Ph. D. grant from Fundação para a Ciência e a Tecnologia (FCT) and whose result is this thesis. Here we classify all generic singularities of the optimal averaged profit for polydynamical systems considering a 2-dimensional parameter*. All types of strategies are studied and the classification is divided in three parts, namely, for stationary strategies, for level cycles and for transitions between these two types of optimal strategies. I recall that all results of this work are related to polydynamical systems.

*Actually, for stationary strategies it is considered a 3-dimensional parameter.
The text above justifies the chapter division of this thesis:

Chapter 1: Preliminary concepts and results. Here we provide all the basic information to make possible the understanding of further results. Important concepts, such as stationary strategies and level cycles, are presented and an important known result is stated.

Chapter 2: Stationary strategies. The main results of this chapter are the classification of all generic singularities:

- of the set consisting of the union of the zeros of all admissible velocities of a polydynamical system
- of the stationary domain for a 3-dimensional parameter
- of the optimal averaged profit for stationary strategies and a 3-dimensional parameter.

Chapter 3: Level cycles. In this chapter we classify all generic singularities of the optimal averaged profit when it is provided by level cycles, for a 2-dimensional parameter.

Chapter 4: Transition between strategies. This chapter contains, for a 2-dimensional parameter, the classification of all generic singularities of the optimal averaged profit at parameter values where it is mandatory to switch between the previous two optimal strategies to obtain the maximum averaged profit.

I dedicate some final words to acknowledgements. I want to express my deep gratitude to my supervisors, A.A. Davydov and H. Mena-Matos, for their personal support, encouragement and helpful comments. I’m very proud to have worked with so special persons during the last six years. I also want to thank FCT and Fundo Social Europeu (FSE) through programs of “III Quadro Comunitário de Apoio” for the Ph. D. grant (SFRH/BD/22308/2005) and other financial support, such as working visits and conferences participation. I also have to thank CMUP for financial support in some conferences participation. Finally, I thank Departamento de Matemática da Universidade do Porto for providing me very good working conditions.
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Assumptions/Simplifications

Smoothness of the objects

Unless otherwise stated, we assume that the original objects treated in this work are smooth, namely, manifolds, vector fields, profit densities and polydynamical systems. However, this assumption can lead to the study of other objects which may not be smooth, for example, the maximum velocity and the optimal averaged profit.

Genericity and Whitney Topology

We introduce in the space of our objects the fine smooth Whitney topology. A property is generic (or holds generically) if it holds for any object belonging to some open everywhere dense subset in this topology.

Families of objects

To study the optimal averaged profit we consider pairs of \( k \)-parameter families of polydynamical systems and of profit densities on the circle when both families have the same parameter space. So, we refer to them as \( k \)-parameter families of pairs of polydynamical systems and profit densities on the circle.

We consider \( k \)-parameter families of objects defined on a phase space such as vector fields, profit densities, polydynamical systems and pairs of these last two objects. We will use the classical identification ([2], p. 245) of one of such families with the map defined on the product space of the phase space by the parameter space, assigning to each point \((x,p)\) the value of the object at \( x \) for the fixed value \( p \) of the parameter.

Fibred spaces and \( \mathcal{F} \)-equivalence

We consider objects defined on product spaces. Each product space \( X \times P \) is regarded as a fibred space over the parameter space \( P \), that is, as the union of all sets of the form \( X \times \{p\} \), with \( p \in P \), which are called fibres. A map from a fibred space to itself is fibred if it preserves the fibration, that is, if it sends fibres to fibres. Note that a fibred map on a fibred space \( X \times P \) over \( P \) has the form \((x,p) \mapsto (\varphi(x,p), h(p))\).
We introduce an equivalence relation for these spaces: two objects of the same nature defined on a fibred space are $\mathcal{F}$-equivalent if one of them can be carried out into the other by a fibred diffeomorphism.

$\Gamma^-, R^+$ and $\mathcal{F}^+$-equivalence

Two germs of functions are $\Gamma$-equivalent if their graphs are $\mathcal{F}$-equivalent, considering the product space of the functions domain by the real axis as a fibred space over the domain. The diffeomorphism carrying one graph into the other takes the form $(p,a) \mapsto (\varphi(p), h(p,a))$ where $p$ and $a$ are variables in the function’s domain and in the real axis, respectively.

$R^+$-equivalence is the particular case of $\Gamma$-equivalence when the second component $h$ of the diffeomorphism is of the form $a + c(p)$, where $c$ is a smooth function. For example, the germ of a smooth function at a point is $R^+$-equivalent to the germ of the zero function at the origin. In fact, if $G$ is a smooth function at a point $p_0$ on a manifold, we consider a coordinate system $\varphi$ with origin at $p_0$ and, finally, the germ of the fibred diffeomorphism $(p,a) \mapsto (\varphi(p), a - G(p))$ at $(p_0,G(p_0))$, to obtain the desirable form.

In Singularity Theory literature it is well known another $R^+$-equivalence, which is defined for germs of families of functions ([6], p.304): two families $F_1$ and $F_2$ are said to be $R^+$-equivalent if one of them is mapped to the other by a suitable fibred diffeomorphism $\Psi$ composed with the addition of a smooth function $\psi$ of the parameter, that is,

$$F_1(x,p) = F_2(h(x,p),\varphi(p)) + \psi(p),$$

where $\Psi(x,p) = (h(x,p),\varphi(p))$. There is an analogous definition for germs.

In order to avoid confusion with these different concepts, in this work we call $\mathcal{F}^+$-equivalence to the $R^+$-equivalence of families of functions.

Simplification of math writing

When considering changes of coordinates we write $\tilde{y} = \phi(y,q)$ to mean that the coordinate $y$ is replaced by a new coordinate $\tilde{y}$. However, after this change of coordinate we omit the tilde to simplify the math writing.

Reference to Mather Division Theorem and Transversality Theorems

We use sistematically three theorems which are basic tools in Singularity Theory, namely, Mather Division Theorem, Thom Transversality Theorem and Multijet Transversality Theorem. They can be found in every classical book on the area, such as [16]. Therefore, to avoid repetition of the cited literature we omit the reference to these three theorems.
Introduction

In this work we classify generic singularities that appear naturally when we introduce a parameter in a well known optimization problem. We start describing this problem.

The optimization problem

A smooth control system on a smooth compact manifold $M$ is a dynamical system defined on $M$ by a smooth vector field $v$ that depends smoothly on a control parameter $u$ belonging to a smooth closed manifold or a disjoint union of such manifolds, $U$, with at least two different points.

$M$ is the phase space and $U$ is the control space of the control system. Each one of the vector fields that is obtained when it is fixed a control parameter value $u$, $v(\cdot, u)$, is called an admissible velocity and we denote it by $v_u$. Given a point $x_0$ on the phase space, the set of admissible velocities at $x_0$ is given by

$$V(x_0) = \{v_u(x_0) : u \in U\}.$$ 

An admissible motion $x : \mathbb{R} \to M$ of the control system is an absolutely continuous map for which the velocity of motion at each time of differentiability, $\dot{x}(t)$, is an admissible velocity, that is, $\dot{x}(t) \in V(x(t))$.

Remark. Because the phase space is compact, any admissible motion of the control system is defined for all $t \in \mathbb{R}$.

When, additionally, there is a profit density on $M$, that is, a smooth function defined on the phase space of the control system, then an admissible motion $x$ provides the averaged profit on the infinite horizon given by

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t))dt.$$ 

Remark. If this limit does not exist one must take its upper limit. To simplify the exposition we assume that this limit always exists.
Finally, the optimization problem is stated as follows:

To maximize the averaged profit on the infinite horizon

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt.
\]

among all admissible motions of the control system.

Such maximum is called the optimal averaged profit and any strategy providing it is called an optimal strategy. This is an important problem of Control Theory already studied in different ways [3], [25], [26].

Example: Consider the following control system on the circle:

\[ \dot{x} = v(x, u), \quad x \in S^1, \quad u \in \{1, 2, 3\}, \]

where \(v_1(x) = \cos x, v_2(x) = 1\) and \(v_3(x) = -1\). Suppose that there is a profit density on the circle given by \(f(x) = \sin(x)\). Observe that the averaged profit on the infinite horizon can never exceed the maximum of the profit density, that is, 1. Then, if we find an admissible motion providing an averaged profit on the infinite horizon equal to 1 then it will be an optimal strategy. If we observe that \(\pi/2\) provides the maximum of the profit density and is an equilibrium point of the first admissible velocity, then we define the admissible motion \(x(t) = \pi/2\) consisting in the permanent staying at \(\pi/2\). So, we conclude that the optimal averaged profit is 1 and that the choice of this motion is an optimal strategy. It is also easy to see [21] that the admissible motion that consists in moving successively from \(\pi/2\) until \(\pi/2 + 1/i\) and then getting back to \(\pi/2\), for every natural \(i\), also provides the optimal averaged profit and so, the choice of this motion is also an optimal strategy. Observe that this motion consists, in the limit, in the permanent staying at the equilibrium point \(\pi/2\).

The Singularity Theory problem

When the optimization problem described above depends on a parameter belonging to a smooth manifold, that is, when both the control system and the profit density depend on a parameter then we obtain a problem depending on a parameter:

To maximize the averaged profit on the infinite horizon

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t), p) dt.
\]

among all admissible motions of the control system.
The averaged profit on the infinite horizon also depends, in a natural way, on the parameter and so, for different parameter values we can have different optimal strategies and different values of the optimal averaged profit. Hence, the **optimal averaged profit** is a function of the parameter. Then, this optimization problem gives rise to a problem in Singularity Theory that consists in classifying singularities of the optimal averaged profit as a function of the parameter. This problem in Singularity Theory was firstly introduced by V.I. Arnold, in 2002 [5]. He studied the particular case when the parameter is 1-dimensional and when both the phase space and the control space are the circle. V.I. Arnold proved that among optimal strategies there can appear two different types of strategies, namely, level cycles and stationary strategies. A **level cycle** is a motion that uses the maximum and minimum admissible velocities when the profit density is not greater and greater, respectively, than a certain constant (level); a **stationary strategy** consists in the permanent staying at an equilibrium point. Some generic singularities of the optimal averaged profit were found in [5].

The classification of all generic singularities of the optimal averaged profit function in Arnold’s model was completed in 2005 by A.A. Davydov and H. Mena-Matos [9], [12]. These authors enlarged the concept of equilibrium point: they define it as a **point of the phase space where the convex hull of the admissible velocities contains the zero velocity** [12]. To classify all generic singularities they proved a crucial result:

*Among optimal strategies there is always a level cycle or a stationary strategy.*

With this result the classification of all generic singularities is reduced to just three cases: singularities for level cycles, for stationary strategies and for transitions between these two kind of optimal strategies.

**Our problem**

In this work we classify all generic singularities of the optimal averaged profit for the case of polydynamical systems and of a $k$-dimensional parameter, with $k \leq 2^*$. A **polydynamical system** is a control system where the control space $U$ has a finite number, $n$, of points and so, the set of admissible velocities at a point $(x,p)$ takes the form

$$V(x,p) = \{v_1(x,p), \ldots, v_n(x,p)\}, \quad n \geq 2.$$  

Note that a polydynamical system is a control system and so, all known results cited here for control systems are also valid for our study on polydynamical systems.

---

*Actually, for stationary strategies it is also considered a 3-dimensional parameter.*
Some results presented in this work are already published, namely, all results concerning stationary strategies and a $k$-dimensional parameter, with $k \leq 3$. These results can be found in [20], [21], [22] and [23].
Chapter 1

Preliminary concepts and results

1.1 Stationary strategies and level cycles

In many applications of optimization problems the stationary strategies are preferable. This case leads to the problem of maximizing the averaged profit on the infinite horizon among motions provided by the permanent staying at equilibrium points. For polydynamical systems, when a parameter value is fixed then, in a generic case, equilibrium points form a discrete set. However, there are open intervals with positive and negative admissible velocities, allowing admissible motions that on the infinite horizon provide the same averaged profit as the permanent staying at the points of such intervals [21]. That motivates the epithet ”stationary strategy” for such motions.

Definition 1.1: Consider the space of families of polydynamical systems on a 1-dimensional manifold. The stationary domain is the set of all points where the convex hull of the admissible velocities contains the zero velocity.

The stationary domain consists of two types of points: equilibrium points (where one of the admissible velocities vanishes) and points where two of the admissible velocities have opposite directions. It is easy to see that a point \((x_0, p_0)\) belongs to this set if and only if, for the fixed value \(p_0\), there is an admissible motion converging to \(x_0\) [21].

Stationary strategies can provide the optimal averaged profit, for example, when for a fixed value of the parameter, the maximum of the corresponding profit density is attained at a point of the stationary domain. However, when this type of strategy does not provide the optimal averaged profit it is natural to ask what type of strategies can provide such profit. In [12], we can find an answer to this question: if the optimal averaged profit is not provided by a
stationary strategy then it has to be provided by an admissible motion providing rotations along the circle. In the rest part of this chapter we talk about this new type of strategies and to simplify the presentation we consider the case without parameter.

**Remark.** Given a control system, the *maximum and minimum velocities* are the vector fields defined on the phase space by $x \mapsto \max\{v(x,u) : u \in U\}$ and $x \mapsto \min\{v(x,u) : u \in U\}$. We denote them by $v_{\text{max}}$ and $v_{\text{min}}$, respectively, and sometimes we call them *extremal velocities*.

An admissible motion providing rotation along the circle just uses velocities with the same direction. By a possible change of orientation on the circle we assume that at each point there is a positive admissible velocity. Among these new type of strategies, the better choice consists in moving as slowly as possible at points where the profit density takes bigger values and as quickly as possible at points where this function takes smaller values [5]. It is clear that we have to use the minimum velocity to move as slowly as possible and to use the maximum velocity to move as quickly as possible. Therefore, the velocity that we have to use has the form

$$v_c(x) = \begin{cases} v_{\text{min}}(x), & f(x) > c \\ v_{\text{max}}(x), & f(x) \leq c \end{cases}, \quad c \in \mathbb{R}.$$ 

**Definition 1.2:** Consider a polydynamical system and a profit density on the circle. Suppose that at each point there is a positive admissible velocity.

1. A motion that uses the maximum and minimum velocities when the profit density is not greater and greater, respectively, than a certain constant $c$ is called a level motion (or $c$-level motion if we want to point out the constant).

2. A value of the profit density is called cyclic if for all nearby values the respective level motions provide rotations along the circle.

3. Given a cyclic value, the respective level motion is called a level cycle.

4. A cyclic value whose level cycle provides the maximum averaged profit on the infinite horizon among all level cycles is called an optimal level. The corresponding level cycle is called an optimal level cycle.
We explain the reason for considering just values of the profit density as cyclic values. In fact, let $M$ and $m$ denote the global maximum and minimum of the profit density, respectively. For $c > M$, the $c$-level motion coincides with the $M$-level motion. Besides, all the cases considered in this work are for profit densities with a finite number of critical points. Therefore, there is a finite number of points having level $m$ and consequently, for $c < m$, the $c$- and the $m$-level motions provide the same averaged profit on the infinite horizon.

The following lemma describes the set of cyclic values.

**Lemma 1.3**: Consider a polydynamical system and a profit density on the circle. The set of cyclic values is either empty or is one of the intervals $[m, M]$ or $[A_s, M]$, where $m$ and $M$ denote the global minimum and maximum of the profit density, respectively, and $A_s$ is the optimal averaged profit for stationary strategies (if it exists).

In fact, if the stationary domain $S$ is an empty set then, obviously, $I = [m, M]$. When $S$ is a nonempty set we note that, admitting that at each point there is a positive velocity,

$$f(x) > A_s \Rightarrow v_{\text{min}}(x) > 0$$

and that, for all value $c$ of the profit density, the $c$-level motion provides rotation along the circle if and only if the minimum velocity is positive in the domain where it is used, that is, in the domain $D_c = \{x : f(x) > c\}$. So, if $c \geq A_s$ then the $c$-level cycle provides rotation along the circle because the minimum velocity is positive in $D_c$. But if $c < A_s$ then the $c$-level motion does not provide rotation along the circle because $D_c$ always contains points where $f = A_s$ and at these points the minimum velocity is not positive. It should be remarked that the $A_s$-level motion provides rotation along the circle but $A_s$ is not cyclic.

**Definition 1.4**: Consider a polydynamical system and a profit density on the circle. Because every level cycle is a periodic motion we define the functions:

1. *Period function*: function that to each cyclic value assigns the period of its level cycle. This function is denoted by $T$. 

![Diagram](image-url)
2. Profit function: function that to each cyclic value \( c \) assigns the profit on the interval \([0, T(c)]\) provided by the \( c \)-level cycle. This function is denoted by \( P \).

3. Averaged Profit function: function that to each cyclic value \( c \) assigns the averaged profit on the interval \([0, T(c)]\) provided by the \( c \)-level cycle. This function is denoted by \( A \).

Remark. For a fixed value of the parameter, the averaged profit provided by a level cycle can only equal the global maximum or the global minimum of the profit density when this is a constant function. Therefore, in a generic case these values can never be optimal levels.

For a given level cycle, the control parameter \( u \) must minimize the velocity \( v(x, u) \) at all points \( x \) where the profit density is bigger than the selected level, and must maximize it at all the other points. So, it is possible to get a level cycle choosing the value of the control parameter \( u \) in the dependence on the position of the point of the phase space. Hence, if \( x(t) \) is a \( c \)-level cycle, its period \( T \) and its profit \( P \) on the interval \([0, T]\) are given by the following expressions

\[
T = \int_0^T dt = \oint_0^T \frac{1}{v_c(x)} dx, \quad P = \int_0^T f(x(t)) dt = \oint_0^T f(x) \frac{f(x)}{v_c(x)} dx
\]

where \( v_c \) is, as defined above, the velocity used along the \( c \)-level cycle. Notice that the averaged profit on the infinite horizon is

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \frac{1}{T(c)} \int_0^{T(c)} f(x(t)) dt = \frac{P(c)}{T(c)} = A(c).
\]

Therefore, an optimal level maximizes the averaged profit function \( A \). We finish this section stating the theorem on the optimality of the stationary strategies and level cycles.

Theorem 1.5 ([12], p.19): For a continuous control system and a continuous profit density on the circle the maximum averaged profit (on the infinite horizon) can always be provided either by a stationary strategy or by a level cycle.

Using this result the classification of all generic singularities of the optimal averaged profit can be reduced to just three cases: for stationary strategies, for level cycles and for transitions between these two kind of strategies.
Chapter 2

Stationary strategies

In this chapter we present the classification of all generic singularities of the optimal averaged profit when in the Singularity Theory problem described on page 4 only stationary strategies are considered, for the case of polydynamical systems and a $k$-dimensional parameter, with $k \leq 3$.

**Definition 2.1:** The optimal averaged profit for stationary strategies is the function that to each parameter value assigns the maximum value of the averaged profit on the infinite horizon among all stationary strategies. This function will be denoted by $A_s$.

It is clear that a stationary strategy at a point of the stationary domain provides an averaged profit on the infinite horizon equal to the value of the family of profit densities at that point. Therefore, the optimal averaged profit $A_s$ for stationary strategies is the solution of the extremal problem with constraints:

$$A_s(p) = \max_{x \in S(p)} f(x, p),$$

where $S(p)$ is the set of all points $x$ on the phase space such that $(x, p)$ belongs to the stationary domain. This profit is well defined for all parameter values $p$ such that $S(p)$ is nonempty because in this case $S(p)$ is a compact set. We say that the profit $A_s(p_0)$ is attained at $(x_0, p_0)$ when $f(x_0, p_0) = A_s(p_0)$.

So, the classification of all generic singularities of the optimal averaged profit for stationary strategies can be done in two steps. Firstly we classify all generic singularities of the stationary domain and then those of the solution of problem (2.1). Due to the stability of the stationary domain of a generic family of polydynamical systems, we fix a stationary domain with generic singularities for one of these families and after that we study the singularities of $A_s$ for a typical family of profit densities [9].
To obtain the classification of all generic singularities of the stationary domain (third section) we begin studying equilibrium points of admissible velocities (first section) and boundary points of the stationary domain (second section) because, in fact, boundary points of the stationary domain are equilibrium points of at least one of the admissible velocities [22].

2.1 Equilibrium points of admissible velocities

In this section we consider $k$-parameter families of vector fields on a 1-dimensional smooth manifold and we classify all generic singularities of the zero level of such families.

**Definition 2.2:** Consider a family $v$ of vector fields on a 1-dimensional manifold. A point $(x_0, p_0)$ of the product space of the phase space by the parameter space is called an equilibrium point of $v$ if $x_0$ is an equilibrium point of the vector field $v(\cdot,p_0)$, that is , $v(x_0,p_0) = 0$.

**Theorem 2.3 ([2]):** Generically, the zero level of a $k$-parameter family of vector fields on a 1-dimensional manifold is either empty or is a $k$-dimensional submanifold of the product space of the phase space by the parameter space.

**Proof:** Consider the space of $k$-parameter families of vector fields on a 1-dimensional manifold. The set of 0-jets of families at equilibrium points is a codimension 1 closed submanifold of the space of 0-jets of such families. Due to Thom Transversality Theorem, in a generic case the pre-image of this submanifold by a 0-jet extension is either empty or is a codimension 1 submanifold of the product space of the phase space by the parameter space, and, consequently, it has dimension $k$.

**Remark.** In this theorem the dimension of the phase space is not essential.

**Definition 2.4:** An equilibrium point of a family of vector fields on a 1-dimensional manifold is called an equilibrium point of type $A_k$ ($k \geq 0$) if the germ of the zero level of the family at that point is $\mathcal{F}$-equivalent to the germ at the origin of the zero level of the family $A_k$ where

$$A_0(x,p) = x \quad \text{and} \quad A_k(x,p) = x^{k+1} + \sum_{i=1}^{k} p_i x^{k-i}, \; k \geq 1$$

where $x$ and $p$ are local coordinates along the phase space and the parameter space, respectively.
2.1. Equilibrium points of admissible velocities

Note that we use the same expression $A_k$ to mention both types of equilibrium points and families of vector fields.

**Remark.** In a fixed local coordinate system we can consider a vector field on a 1-dimensional manifold as a function. However, $\mathcal{F}$-equivalence acts differently on a family of vector fields and on the corresponding family of functions. In fact,

1. two germs $(v_1, (x_0, p_0))$ and $(v_2, (y_0, q_0))$ of families of functions are $\mathcal{F}$-equivalent if there is a diffeomorphism $\phi$ of the form $\phi(x, p) = (\varphi(x, p), h(p))$ such that $\phi(x_0, p_0) = (y_0, q_0)$ and, in a neighborhood of $(x_0, p_0)$, $v_2 \circ \phi = v_1$.

2. two germs $(v_1, (x_0, p_0))$ and $(v_2, (y_0, q_0))$ of families of vector fields are $\mathcal{F}$-equivalent if there is a diffeomorphism $\phi$ of the form $\phi(x, p) = (\varphi(x, p), h(p))$ such that $\phi(x_0, p_0) = (y_0, q_0)$ and, in a neighborhood of $(x_0, p_0)$, $v_2 \circ \phi = \frac{\partial \varphi}{\partial x} \cdot v_1$.

Therefore, if $(v_1, (x_0, p_0))$ and $(v_2, (y_0, q_0))$ are $\mathcal{F}$-equivalent as families of functions (respectively, vector fields) then $(V \cdot v_1, (x_0, p_0))$ and $(v_2, (y_0, q_0))$ are $\mathcal{F}$-equivalent as families of vector fields (respectively, functions), for an appropriate function $V$ that does not vanish at the origin. In particular, $\mathcal{F}$-equivalence preserves the zero levels of the families.

**Proposition 2.5:** Consider the space of $k$-parameter families of vector fields on a 1-dimensional manifold. Generically, the germ of a family at an equilibrium point is $\mathcal{F}$-equivalent to the germ at the origin of one of the manifolds $A_l \cdot V$, for some smooth function $V$ that does not vanish at the origin and $0 \leq l \leq k$. In particular, in a generic case every equilibrium point of a $k$-parameter family of vector fields on a 1-dimensional manifold is a point of one of the types $A_l$ with $0 \leq l \leq k$.

**Proof:** As it was said above, in a fixed local coordinate system we can consider a vector field as a function. $\mathcal{F}$-equivalence acts differently on the field and on the corresponding function but preserves their zero levels. But in a generic case the germ of a $k$-parameter family of smooth functions on the line at a zero is $\mathcal{F}$-equivalent to the germ at the origin of either $x$ or $x^{l+1} + p_1 x^{l-1} + \cdots + p_l$, $1 \leq l \leq k$ [6]. Consequently, in a generic case the germ of family at an equilibrium point is $\mathcal{F}$-equivalent to the germ at the origin of $A_l \cdot V$, for some smooth function $V$ that does not vanish at the origin and $0 \leq l \leq k$. The second part of this proposition is now obvious.
2. Stationary strategies

2.2 Boundary points of the stationary domain

In this section we consider \( k \)-parameter families of polydynamical systems on a 1-dimensional smooth manifold and we classify all generic singularities of the zero level of such families. Here we generalize for any natural \( k \) some results in \([21],[22]\) concerning the generic singularities of the stationary domain. Although the proofs are based on the same ideas, they become more complex in what concerns transversality conditions. Besides, this generalization involves a heavier framework on the statement of the results. The main result presented here is already published \([23]\) and it is applied in the next section to obtain the classification of all generic singularities of the stationary domain when \( k \leq 3 \).

Lemma 2.6: Generically, the zero levels of two given admissible velocities are transversal.

Proof: The set of 0-jets of pairs of admissible velocities at common equilibrium points is a codimension 2 closed submanifold of the space of 0-jets of such pairs. Due to Thom Transversality Theorem, the set of pairs whose 0-jets extensions are transversal to this submanifold is open and everywhere dense. In this case, transversality means that at the common equilibrium points of any admissible velocities \( v \) and \( w \) the matrix

\[
\begin{pmatrix}
    v_x & v_{p_1} & \cdots & v_{p_k} \\
    w_x & w_{p_1} & \cdots & w_{p_k}
\end{pmatrix}
\]

has maximum rank, where \( x \) and \( (p_1,\ldots,p_k) \) are local coordinates along the phase space and the parameter space, respectively. Then, in a generic case, the zero levels of two admissible velocities at a common equilibrium point are transversal. \( \blacksquare \)

Definition 2.7: Consider a family of polydynamical systems on a 1-dimensional manifold and denote by \( Z \) the union of the zeros of all admissible velocities on the product space. A point of this set is called a point of type \( A_{I_j} \) with \( I_j = (i_1,\ldots,i_j) \), \( (\text{all } j,i_1,\ldots,i_j \text{ are nonnegative integers and } 0 \leq i_1 \leq \cdots \leq i_j) \), if at this point the germ of the set \( Z \) is \( \mathcal{F} \)-equivalent to the germ at the origin of the set

\[
\left( x^{i_1+1} + \sum_{l=1}^{i_1} p_l x^{i_1-l} \right) \prod_{l=2}^{j} \left( x^{i_l+1} + \sum_{m=|I_l|-i_l}^{|I_l|} p_m x^{|I_l|-m} \right) = 0
\]

where \( x \) and \( p_1, p_2, \ldots \) are local coordinates along the phase space and the parameter space, respectively, \( I_l = (i_1,\ldots,i_l), 1 \leq l \leq j \) and \( |I_l| = l - 1 + i_1 + \cdots + i_l \).

Remark. Note that, in a generic case, a point of type \( A_{I_j} \) is an equilibrium point of exactly \( j \) admissible velocities, which is a point of type \( A_{i_1},\ldots,A_{i_j} \) for each one of the velocities, respectively (Lemma 2.6).
Theorem 2.8: Generically, every point of the set $Z$ of a $k$-parameter family of polydynamical systems on a 1-dimensional manifold is a point of type $A_{I_j}$ with $|I_j| \leq k$.

Proof: For a $k$-parameter family of polydynamical systems with $n$ admissible velocities, generically any point of the set $Z$ is an equilibrium point of exactly $j$ admissible velocities $v_{\alpha_1}, \ldots, v_{\alpha_j}$, where $j, \alpha_1, \ldots, \alpha_j$ are natural numbers not greater than $n$. Due to Proposition 2.5, generically this point is of type $A_{i_1}$ for $v_{\alpha_1}, \ldots, A_{i_j}$ for $v_{\alpha_j}$, where $i_1, \ldots, i_j$ are nonnegative integers. We can always suppose that $0 \leq i_1 \leq \cdots \leq i_j$. Let $I_j = (i_1, \ldots, i_j)$.

We start proving that generically $|I_j| \leq k$. The set of $k$-jets of families at this kind of points is a closed submanifold of the space of $k$-jets and has codimension $|I_j| + 1$. Due to Thom Transversality Theorem, if $|I_j| + 1 > k + 1$ then generically such points do not appear. Due to Proposition 2.5, generically we can consider a fibred local coordinate system around this point where the zero level of one of the admissible velocities takes the form $A_{i_1} = 0$. Thus, applying Mather Division Theorem, in that coordinate system the zero levels of all the others admissible velocities are written in the form

$$x^{i_1+1} + \sum_{m=|I_j|-i_1}^{|I_j|} r_m(p)x^{|I_j|-m} = 0, \quad 2 \leq l \leq j$$

where all $r_m$ are smooth functions vanishing at the origin. Finally, Thom Transversality Theorem implies that the map

$$p \mapsto (p_1, \ldots, p_{i_1+1}(p), \ldots, r_{|I_j|}(p), p_{|I_j|+1}, \ldots, p_k)$$

has maximum rank at the origin. Consequently, selecting new coordinates

$$p_l = r_l(p), \quad i_1 + 1 \leq l \leq |I_j|$$

we get the needed normal form for the set $Z$ near the point under consideration.

As a particular case, we conclude that in a generic case there are no equilibrium points which are common to more than $k + 1$ admissible velocities. Besides, when $k = 2$ only six generic situations may occur: $A_0, A_1, A_2, A_{0,0}, A_{0,1}, A_{0,0,0}$; when $k = 3$, there are five generic situations more: $A_3, A_{0,2}, A_{1,1}, A_{0,0,1}$ and $A_{0,0,0,0}$.

Remark. Note that the change of coordinates on the phase space presented in this theorem is of the form $x + \alpha(p)$, where $\alpha$ is a smooth function. In fact, this coordinate is changed a unique time in the whole proof, namely, to write the zero level of one of the admissible velocities, $v$, in the form $A_{i_1} = 0$, that is, in the form

$$x^{i_1+1} + p_1x^{i_1-1} + \cdots + p_{i_1} = 0$$

(1)
Due to Mather Division Theorem, before we do this change of coordinates, the zero level of $v$ can be written as

$$(x - x_0)^{i_1+1} + \alpha_1(p)(x - x_0)^{i_1} + \cdots + \alpha_{i_1}(p)(x - x_0) + \alpha_{i_1+1}(p) = 0,$$

or

$$(x - x_0 + \frac{\alpha_1(p)}{i_1+1})^{i_1+1} + \sum_{j=0}^{i_1-1} \left( \alpha_{i_1+1-j}(p) - C_j^{i_1+1} \left( \frac{\alpha_1(p)}{i_1+1} \right)^{i_1+1-j} \right)(x - x_0)^j = 0, \quad (2)$$

where all $\alpha_j$ are smooth functions vanishing at $p_0$. Then, comparing expressions (1) and (2) we have that the new coordinate takes the form $x - x_0 + \frac{\alpha_1(p)}{i_1+1}$.

### 2.3 Singularities of the stationary domain

In this section we consider $k$-parameter families of polydynamical systems on a 1-dimensional manifold and use the main result of the previous section to obtain all generic singularities of the stationary domain when $k \leq 3$. The result presented here is already published [23].

**Theorem 2.9**: The germ of the stationary domain of a generic $k$-parameter family of polydynamical systems on a 1-dimensional manifold, at any point is, up to $\mathcal{F}$-equivalence, the germ at the origin of one of the sets from the second column of:

- Table 2.1, if $k = 1$.
- Tables 2.1 and 2.2, if $k = 2$.
- Tables 2.1, 2.2 and 2.3, if $k = 3$.

Moreover, the germs of the stationary domains of a generic family and of any other sufficiently close to it can be reduced one to another by $\mathcal{F}$-equivalence close to the identity.

In these tables, the third and the fourth columns show the type of the point and the number of admissible velocities of the polydynamical system, respectively. The last column points out the codimension in the product space of the phase space by the parameter space of the stratum of the respective singularity.

Note that Tables 2.2 and 2.3 correspond to singularities of the stationary domain at points of type $A_{I_j}$ with $|I_j| = 2$ and 3, respectively. In a natural way, for every $k > 1$, any Table 2.$k$ contains all generic singularities of the stationary domain at points type $A_{I_j}$ with $|I_j| = k$. 
2.3. Singularities of the stationary domain

Table 2.1:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Type</th>
<th>n</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{R}^{k+1}$</td>
<td>Interior</td>
<td>$\geq 2$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$x \leq 0$</td>
<td>$A_0$</td>
<td>$\geq 2$</td>
<td>$\pm 1$</td>
</tr>
<tr>
<td>3±</td>
<td>$\pm (x^2 + p_1) \leq 0$</td>
<td>$A_1$</td>
<td>$\geq 2$</td>
<td>2</td>
</tr>
<tr>
<td>4±</td>
<td>$\pm (x + p_1) \leq 0$</td>
<td>$A_{0,0}$</td>
<td>2</td>
<td>$\geq 3$</td>
</tr>
<tr>
<td>5±</td>
<td>$x \leq 0 \lor \pm (x + p_1) \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Type</th>
<th>n</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$x^3 + p_1 x + p_2 \leq 0$</td>
<td>$A_2$</td>
<td>$\geq 2$</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>$x(x^2 + p_1 x + p_2) \leq 0$</td>
<td>$A_{0,1}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>8±</td>
<td>$x \leq 0 \lor \pm (x^2 + p_1 x + p_2) \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9±</td>
<td>$x(x + p_1) \leq 0 \lor x(x + p_2) \leq 0$</td>
<td>$A_{0,0,0}$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>9±</td>
<td>$\pm x(x + p_1) \leq 0 \lor x(x + p_2) \geq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10±</td>
<td>$x \leq 0 \lor x + p_1 \leq 0 \lor \pm (x + p_2) \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Type</th>
<th>n</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>11±</td>
<td>$\pm (x^3 + p_1 x^2 + p_2 x + p_3) \leq 0$</td>
<td>$A_3$</td>
<td>$\geq 2$</td>
<td>4</td>
</tr>
<tr>
<td>12±</td>
<td>$\pm (x(x^3 + p_1 x^2 + p_2 x + p_3) \leq 0$</td>
<td>$A_{0,2}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>13±</td>
<td>$x \leq 0 \lor \pm (x^3 + p_1 x^2 + p_2 x + p_3) \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14±</td>
<td>$\pm (x^2 + p_1)(x^2 + p_2 x + p_3) \leq 0$</td>
<td>$A_{1,1}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>15±</td>
<td>$x^2 + p_1 \leq 0 \lor x^2 + p_2 x + p_3 \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15±</td>
<td>$\pm (x^2 + p_1) \leq 0 \lor x^2 + p_2 x + p_3 \geq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16±</td>
<td>$\pm x(x + p_1) \leq 0 \lor x(x^2 + p_2 x + p_3) \leq 0$</td>
<td>$A_{0,0,1}$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>17±±</td>
<td>$x \leq 0 \lor \pm (x + p_1) \leq 0 \lor \pm (x^2 + p_2 x + p_3) \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18±±</td>
<td>$x(x + p_1) \leq 0 \lor \pm (x + p_2) \leq 0 \lor \pm (x + p_3) \leq 0$</td>
<td>$A_{0,0,0}$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>19±</td>
<td>$x \leq 0 \lor x + p_1 \leq 0 \lor x + p_2 \leq 0 \lor x + p_3 \leq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19±</td>
<td>$x \leq 0 \lor x + p_1 \leq 0 \lor \pm (x + p_2) \leq 0 \lor \pm x + p_3 \geq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof: Consider a $k$-parameter family of polydynamical systems on the circle and a point of the stationary domain. Obviously, the germ of the stationary domain at an interior point is $\mathcal{F}$-equivalent to the germ of $\mathbb{R}^{k+1}$ at the origin. Consider, now, a point on the boundary of the set. At this point at least one of the admissible velocities has to vanish and all the other that do not vanish have the same direction [22].
1. When there is exactly one admissible velocity vanishing then, in a generic case, there is a fibred local coordinate system around that point where the vanishing velocity can be written as $A_l \cdot V$, where $0 \leq l \leq k$ and $V$ is a smooth function that does not vanish at the origin (Proposition 2.5). That leads to the singularities 2, 3±, 6 and 11± of Tables 2.1, 2.2 and 2.3. Note that the following pairs of sets are $\mathcal{F}$-equivalent:

- $x \leq 0$ and $x \geq 0$,
- $x^3 + p_1x + p_2 \leq 0$ and $x^3 + p_1x + p_2 \geq 0$.

2. When there are more than one, say $j$, admissible velocities vanishing then all singularities are obtained as an immediate consequence of Theorem 2.8 and by the following reasoning: if the polydynamical system just consists of these $j$ velocities then, in some fibred local coordinate system, the stationary domain is the set of points where at least two admissible velocities have opposite directions; if there are more admissible velocities, then all nonvanishing velocities at that point have the same direction and so, we must include in the former stationary domain the region where the $j$ vanishing velocities have opposite direction to that one of the nonvanishing velocities.

The stability of the stationary domain up to small perturbations of a generic family of polydynamical systems follows immediately from the transversality conditions used to conclude the genericity of Theorem 2.8.

2.4 Singularities of the optimal averaged profit for stationary strategies

Here we list all generic singularities of the optimal averaged profit for stationary strategies. We start presenting an example to motivate Definition 2.10.

Example: Consider a 1-parameter family $\{(v_1, v_2), f\}$ of polydynamical systems and profit densities on the circle where:

- $v_1(x, p) = 4\cos^2 x + 4p^2 - 1$,
- $v_2(x, p) = \sin x + \sqrt{2}/2$,
- $f(x, p) = \sin (x + \pi/4)$.

In the figure on the right the stationary domain is painted in grey and the points providing the optimal profit $A_s$ are marked with a dark line. It is clear that the study of $A_s$ is done in two different ways:
1. at \( p = \pm \frac{1}{2} \) we have to look at the family of profit densities and at the stationary domain in a neighborhood of two points, namely, \((\frac{\pi}{2}, \pm \frac{1}{2})\) and \((\frac{7\pi}{4}, \pm \frac{1}{2})\).

2. at any other parameter value it is just necessary to look at those families in a neighborhood of a unique point.

**Definition 2.10:** Consider a family of pairs of polydynamical systems and profit densities on a 1-dimensional manifold. Let \( S \) be the stationary domain and \( S^* \) be the subset of \( S \) whose points provide the optimal profit \( A_s \), that is, \( S^* = \{(x,p) \in S : f(x,p) = A_s(p)\} \). A parameter value \( p \) is a value without competition if the closure of \( S^* \) has a unique point of the form \((x,p)\). If there exists more than one point of that form, \( p \) is a value with competition and those points are said to be competing to provide the profit \( A_s(p) \).

To study singularities of the profit \( A_s \) at values without competition it is sufficient to look at the stationary domain and at the family of profit densities in a neighborhood of the point where such profit is attained. However, the study of singularities of this function at values with competition demands the consideration of neighborhoods of more than one point of the stationary domain. This fact justifies calling point singularities to singularities of the first type, and competition singularities to the others.

Therefore, the classification of all generic singularities of the optimal profit \( A_s \) can be done in two parts: first, we study point singularities; after that we analyse competition singularities.

### 2.4.1 Point singularities

In this subsection we classify all generic singularities of the optimal profit \( A_s \) at values without competition. The following definition gives us the possibility to simplify the further exposition.

**Definition 2.11:** Consider a family \((V,f)\) of pairs of polydynamical systems and profit densities on a 1-dimensional manifold. A point of type \( A_{I_j} \) of the stationary domain is called a point of type \( A_{I_j}^i \) if \( i \) is the smallest natural such that the derivative \( \frac{\partial^i f}{\partial x^i} \) does not vanish at that point, where \( x \) is a coordinate along the circle. An interior point of the stationary domain is called a point of type \( I^i \) if \( i \) is defined in the same manner.

**Lemma 2.12:** Consider a \( k \)-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional manifold \( M \), when the family of polydynamical systems is generic and \( k \leq 3 \). Suppose that the stationary domain at a point of type \( A_{I_j}^i \) or \( I^i \) has a codimension \( m \) singularity of Tables 2.1-2.3. Then, generically,

\[
i + m \leq k + 2.
\]
Proof: Consider a point of type $A_i^j$ or $I^j$ where the stationary domain has a codimension $m$ singularity of Tables 2.1-2.3. At such a point, a family of pairs of polydynamical systems and profit densities satisfies exactly $\mu$ independent equalities (among other conditions) with $\mu = m + i - 1$. Consider the subset $Y$ of $M \times P$ consisting of the points at which just these equalities are satisfied. In the jet space $J^5(M \times P, \mathbb{R})^*$, the set $W$ of jets of families at points of $Y$ is a codimension $\mu$ closed submanifold. Due to Thom Transversality Theorem, for a generic family $(V, f)$, the jet extension $j^5(V, f)$ is transversal to $W$ and, therefore, $(j^5(V, f))^{-1}(W)$ is either empty or is a codimension $\mu$ submanifold of $M \times P$. But $(j^5(V, f))^{-1}(W) = Y$ and $Y$ is a nonempty set. Therefore, generically this codimension can not be greater than the dimension $k + 1$ of $M \times P$, and so, $i + m \leq k + 2$.

Corollary 2.13: Generically, the optimal averaged profit for stationary strategies can be attained at points of type:

- $I^2$, $A_0^1$, $A_1^1$, $A_0,0^1$, $A_0^2$ if $k = 1$, or else
- $I^4$, $A_2^1$, $A_0,0^1$, $A_0,0,0^1$, $A_1^2$, $A_0,0^2$, $A_0^3$ if $k = 2$, or else
- $A_3^1$, $A_0,2^1$, $A_1,1^1$, $A_0,0,0,0^1$, $A_2^2$, $A_0,1^2$, $A_0,0,0,0^2$, $A_1^3$, $A_0,0,0^3$, $A_0^4$, if $k = 3$.

Proof: For $k = 1$, due to the previous lemma we have $i + m \leq 3$. Then, if the profit $A_s$ is attained at an interior point of the stationary domain $S$ then $m = 0$ and $i$ is even because such point provides a maximum of the family of profit densities. So, we obtain points of type $I^2$. If the profit is attained at a boundary point of $S$ then $i \geq 1$ and, looking at Tables 2.1-2.3, we see that $m \geq 1$ and conclude that there are only three possible values for the pair $(m, i)$, namely, $(1, 1)$, $(1, 2)$ and $(2, 1)$. The first pair leads to points of type $A_0^1$; the second leads to points of type $A_0^2$ and the third leads to points of types $A_1^1$ and $A_0,0^1$. For $k = 2$ and $k = 3$ the process is the same.

Theorem 2.14: For a generic $k$-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional manifold, the germ of the optimal profit for stationary strategies at a parameter value without competition is, up to $R^+\text{-equivalence}$, the germ at the origin of one of the functions from the second column of:

- Table 2.4, if $k = 1$;
- Tables 2.4 and 2.5, if $k = 2$;
- Tables 2.4, 2.5 and 2.6, if $k = 3$.

*Using Thom Transversality Theorem, it is easy to prove that in a generic case $i \leq k + 2 \leq 5$, for $k \leq 3$. 

2.4. Singularities of the optimal averaged profit for stationary strategies

On Tables 2.4, 2.5 and 2.6, columns 3 and 4 describe the type of point leading to the optimal profit and the codimension of these singularities in the parameter space, respectively.

Table 2.4:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Point</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$I^2$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$p_1</td>
<td>p_1</td>
<td>$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{p_1}$</td>
<td>$A_1^1$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$</td>
<td>p_1</td>
<td>$</td>
</tr>
</tbody>
</table>

Table 2.5:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Point</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$\max{-x^4 + p_1 x^2 + p_2 x : x \in \mathbb{R}}$</td>
<td>$I^4$</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$\max{x^3 + p_1 x^2 + p_2 x : x \leq 0}$</td>
<td>$A_0^3$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\pm\max{-x^2 + p_1 x : x(x + p_2) \leq 0}$</td>
<td>$A_1^2$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\sqrt{p_1}</td>
<td>p_2</td>
<td>$</td>
</tr>
<tr>
<td>9</td>
<td>$</td>
<td>p_1 p_2</td>
<td>$</td>
</tr>
<tr>
<td>10</td>
<td>$\max{-x^2 + p_1 x : x(x + p_2) \leq 0}$</td>
<td>$A_2^1$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$\max{-x^2 + p_1 x : x \leq \max{0, p_2}}$</td>
<td>$A_0,1^1$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\max{x : x^3 + p_1 x + p_2 = 0}$</td>
<td>$A_0,0,0^1$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$\max{\sqrt{p_1}, p_2}$</td>
<td>$A_0,0,1^1$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$\max{</td>
<td>p_1</td>
<td>, p_2}$</td>
</tr>
</tbody>
</table>

Some singularities listed in these tables are well known in Singularity Theory and appear in many other classifications [5], [7], [8], [10], [15].

**Proof**: Consider a parameter value $p_0$ whose profit $A_s(p_0)$ is attained at a unique point $(x_0, p_0)$ of the stationary domain $S$. We select this point as the origin. Due to Corollary 2.13, in a generic case we have to consider the following situations:

**Situation 1.** The origin is a point of type $I^2$

The Implicit Function Theorem implies that near the origin equation $f_x(x, p) = 0$ is equivalent to $x = X(p)$, for some smooth function $X$ vanishing at the origin. Then, for every $p$ near $p = 0$, $A_s(p) = f(X(p), p)$ and so, $A_s$ is a smooth function near $p = 0$. Therefore, we obtain singularity 1.
2. Stationary strategies

Table 2.6:

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Point</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$\max{-x^4 + p_1 x^3 + p_2 x^2 + p_3 x : x \leq 0}$</td>
<td>$A_0^4$</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>$\max{x^3 + p_1 x^2 + p_2 x : x^2 + p_3 \leq 0}$</td>
<td>$A_1^3$</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>$\max{x^3 + p_1 x^2 + p_2 x : x(x + p_3) \leq 0}$</td>
<td>$A_{0,0}^3$</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>$\max{x^3 + p_1 x^2 + p_2 x : x \leq \max{0, p_3}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>$\max{-x^2 + p_1 x : x^3 + p_2 x + p_3 \leq 0}$</td>
<td>$A_2^2$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>$\max{-x^2 + p_1 x : x(x^2 + p_2 x + p_3) \leq 0}$</td>
<td>$A_{0,1}^2$</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>$\max{-x^2 + p_1 x : x \leq 0 \lor \pm(x^2 + p_2 x + p_3) \leq 0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>$\max{p_1^2, p_2^2, p_3^2}$</td>
<td>$A_{0,0,0}^2$</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>$\max{-x^2 + p_1 x : x(x + p_3) \leq 0 \lor x(x + p_3) \leq 0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>$\max{-x^2 + p_1 x : x \leq \max{0, p_2, p_3}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>$\max{x : x^4 + p_1 x^2 + p_2 x + p_3 = 0}$</td>
<td>$A_3^1$</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>$\max{x : x(x^3 + p_1 x^2 + p_2 x + p_3) = 0}$</td>
<td>$A_{0,2}^1$</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>$\max{\sqrt{p_1}, \sqrt{p_2} + p_3}$</td>
<td>$A_{1,1}^1$</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2} + p_3}$</td>
</tr>
<tr>
<td>29</td>
<td>$\max{</td>
<td>p_1</td>
<td>,</td>
</tr>
</tbody>
</table>

**Situation 2.** The origin is a point of type $I^4$

By classical results on Singularity Theory [6] and transversality theorems we immediately obtain singularity 5: $\max\{-x^4 + p_1 x^2 + p_2 x : x \in \mathbb{R}\}$.

In the rest part of the proof the origin is a point on the boundary of the stationary domain, that is, is a point of type $A_{I^4}$. When $i = 1$ the proof is very simple. In fact, by classical results on Singularity Theory [6] we know that the germs of $f$ and of $x$ at the origin are $\mathcal{F}^+$-equivalent. Therefore, when $f$ is restricted to the stationary domain it attains its maximum on the boundary of the set due to the absence of local maxima. On page 16 we have seen that there is a change of coordinates preserving the previous normal form for $f$ up to $R^+$-equivalence and where the stationary domain has one of the normal forms listed on Theorem 2.8. Thus,

**Situation 3.** The origin is a point of type $A_0^1$

The germs at the origin of $A_4$ and of $\max\{x : x = 0\} = 0$ are $R^+$-equivalent. We obtain singularity 1.

**Situation 4.** The origin is a point of type $A_1^1$

The germs at the origin of $A_4$ and of $\max\{x : x^2 + p_1 = 0\} = \max\{\pm\sqrt{-p_1}\} = \sqrt{-p_1}$ are $R^+$-equivalent. Changing the sign of $p_1$ we obtain singularity 3.
2.4. Singularities of the optimal averaged profit for stationary strategies

Situation 5. The origin is a point of type \( A_{0,0}^1 \)

The germs at the origin of \( A_s \) and of \( \max\{x : x(x + p_1) = 0\} = \max\{0, -p_1\} \) are \( R^+ \)-equivalent. Adding \( \frac{p_1}{2} \) to this function and then doing \( \tilde{p}_1 = \frac{p_1}{2} \) we get singularity 4.

Situation 6. The origin is a point of type \( A_2^1 \)

We immediately obtain singularity 12: \( \max\{x : x^3 + p_1x + p_2 = 0\} \).

Situation 7. The origin is a point of type \( A_{0,1}^1 \)

The germs at the origin of \( A_s \) and of \( \max\{x : x(x^2 + p_1x + p_2) = 0\} \) are \( R^+ \)-equivalent. Considering the change of coordinates \( \tilde{x} = x + \frac{p_1}{2}, \tilde{p}_1 = \frac{p_1^2}{4} - p_2, \tilde{p}_2 = -\frac{p_1}{2} \) we conclude that the germs at the origin of \( A_s \) and of \( \max\{x : (x^2 - p_1)(x - p_2) = 0\} = \max\{\pm \sqrt{p_1}, p_2\} = \max\{\sqrt{p_1}, p_2\}, p_1 \geq 0, \) are \( R^+ \)-equivalent. We obtain singularity 13.

Situation 8. The origin is a point of type \( A_{0,0,0}^1 \)

The germs at the origin of \( A_s \) and of \( \max\{x : x(x + p_1)(x + p_2) = 0\} = \max\{0, -p_1, -p_2\} \) are \( R^+ \)-equivalent. Adding \( \frac{p_1}{2} \) to this function and then considering \( \tilde{p}_1 = \frac{p_1}{2}, \tilde{p}_2 = -p_2 + \frac{p_1}{2} \) we obtain singularity 14.

Situation 9. The origin is a point of type \( A_3^1 \)

We immediately obtain singularity 25: \( \max\{x : x^4 + p_1x^2 + p_2x + p_3 = 0\} \).

Situation 10. The origin is a point of type \( A_{0,2}^1 \)

We immediately obtain singularity 26: \( \max\{x : x(x^3 + p_1x^2 + p_2x + p_3) = 0\} \).

Situation 11. The origin is a point of type \( A_{1,1}^1 \)

The germs at the origin of \( A_s \) and of \( \max\{x : (x^2 + p_1)(x^2 + p_2x + p_3) = 0\} \) are \( R^+ \)-equivalent. Considering new coordinates \( \tilde{p}_1 = -p_1, \tilde{p}_2 = \frac{p_2^2}{4} - p_3, \tilde{p}_3 = -\frac{p_2}{2} \) we conclude that the germs at the origin of \( A_s \) and of \( \max\{x : (x^2 - p_1)(x - p_3)^2 - p_2 = 0\} = \max\{\pm \sqrt{p_1}, \sqrt{p_2} + p_3\} = \max\{\sqrt{p_1}, \sqrt{p_2} + p_3\}, \) with \( p_1, p_2 \geq 0 \) are \( R^+ \)-equivalent. We obtain singularity 27.

Situation 12. The origin is a point of type \( A_{0,0,1}^1 \)

The germs at the origin of \( A_s \) and of \( \max\{x : x(x + p_1)(x^2 + p_2x + p_3) = 0\} \) are \( R^+ \)-equivalent. Doing \( \tilde{x} = x + \frac{p_1}{2} \), \( \tilde{p}_1 = \frac{p_1}{2}, \tilde{p}_2 = -p_3 + \frac{p_2^2}{4}, \tilde{p}_3 = \frac{p_1}{2} - p_2 \) we conclude that the germs at the origin of \( A_s \) and of \( \max\{x : (x^2 - p_1^2)(x - p_3)^2 - p_2 = 0\} = \max\{|p_1|, \sqrt{p_2} + p_3\} = \max\{|p_1|, \sqrt{p_2} + p_3\}, p_2 \geq 0, \) are \( R^+ \)-equivalent. We obtain singularity 28.
Situation 13. The origin is a point of type $A_{0,0,0,0}$.\(^1\)

The germs at the origin of $A_s$ and of $\max\{x : x(x + p_1)(x + p_2)(x + p_3) = 0\} = \max\{0, -p_1, -p_2, -p_3\}$ are $R^+$-equivalent. Adding $\frac{p_1}{2}$ to this function and then considering new coordinates $\tilde{p}_1 = \frac{p_1}{2}, \tilde{p}_2 = \frac{p_2 - p_3}{2}, \tilde{p}_3 = \frac{p_1 - p_2 - p_3}{2}$ we obtain singularity 29.

To prove all the other situations we carry out the following steps:

1.) find a normal form for the family of profit densities $f$ up to $\mathcal{F}^+$-equivalence using classical results on Singularity Theory;
2.) choose new coordinates to write the boundary of the stationary domain in the normal form presented in Tables 2.1-2.3 (note that this choice of new coordinates can change the normal form of the density);
3.) use Theorem 2.9 to identify all possible normal forms for the stationary domain; here, the fact that $f(0, 0)$ corresponds to a maximum of $f(\cdot, 0)$ can be used to exclude some of these normal forms;
4.) compare the local maxima attained on the interior of the stationary domain and the value of the density on its boundary to obtain the optimal averaged profit.

Situation 14. The origin is a point of type $A_0^2$.

In this situation there are two conditions, namely, $f_x = 0$ and $v = 0$ at the origin, for some admissible velocity $v$. By classical results on Singularity Theory [6] we know that generically at the origin the germ of $f$ is $\mathcal{F}^+$-equivalent to the germ of one of the functions $\pm x^2$.

The boundary of the stationary domain is given by $v = 0$. By Mather Division Theorem, around the origin equation $v = 0$ is equivalent to $x + r(p) = 0$, for some smooth function $r$ vanishing at the origin. Transversality theorems imply that generically the matrix

$$\begin{pmatrix}
 f_{xx} & f_{xp_1} & f_{xp_2} & f_{xp_3} \\
 v_x & v_{p_1} & v_{p_2} & v_{p_3}
\end{pmatrix}(0)
$$

has maximum rank. Then, without loss of generality, we can suppose that $r_{p_1}(0) \neq 0$. Considering a new coordinate $\tilde{p}_1 = -r(p)$, the normal form for $f$ is preserved and the boundary of the stationary domain takes the form $x = p_1$. We can always assume that $S = \{x \leq p_1\}$, otherwise we consider $\tilde{x} = -x$ and $\tilde{p}_1 = -p_1$. Then, because $f(0, 0)$ corresponds to a maximum of $f(\cdot, 0)$, the normal form for $f$ must be $-x^2$. So, the germs at the origin of $A_s$ and of $\max\{-p_1^2; 0, p_1 \geq 0\}$ are $R^+$-equivalent. Adding $\frac{p_1^2}{2}$ to this function and then considering a new coordinate $\tilde{p}_1 = \frac{\sqrt{2}}{2}p_1$ we obtain singularity 2.
**Situation 15.** The origin is a point of type $A_1^2$

In this situation there are three conditions, namely, $f_x = 0$ and $v = v_x = 0$ at the origin, for some admissible velocity $v$. As in the previous situation, at the origin the germ of $f$ is $\mathcal{F}^+$-equivalent to the germ of one of the functions $\pm x^2$.

The boundary of the stationary domain is given by $v = 0$. By Mather Division Theorem, around the origin equation $v = 0$ is equivalent to $x^2 + r_1(p)x + r_2(p) = 0$, for some smooth functions $r_1$ and $r_2$ vanishing at the origin. Transversality theorems imply that generically the matrix

$$
\begin{pmatrix}
  r_{1,p_1} & r_{1,p_2} & r_{1,p_3} \\
  r_{2,p_1} & r_{2,p_2} & r_{2,p_3}
\end{pmatrix}
(0)
$$

has maximum rank. Then, without loss of generality, we can choose $\tilde{p}_1 = r_1(p)$ and $\tilde{p}_2 = r_2(p)$ as new coordinates, preserving the normal form for $f$ and writing the boundary of the stationary domain as $x^2 + p_1 x + p_2 = 0$. Now, considering $\tilde{x} = x + p_1/2$, $\tilde{p}_1 = \mp p_1/2$ and $\tilde{p}_2 = p_2 - (p_1/2)^2$ we obtain that $f$ is $\mathcal{F}^+$-equivalent to $\pm x^2 + p_1 x$ and that the boundary of the stationary domain is given by $x^2 + p_2 = 0$. By Theorem 2.9 we have to consider the following two normal forms for the stationary domain $S$: $\{ \pm (x^2 + p_2) \leq 0 \}$.

1. If the normal form for $f$ is $x^2 + p_1 x$ then, because $f(0,0)$ corresponds to a maximum of $f(\cdot,0)$, we just have to consider $S = \{ x^2 + p_2 \leq 0 \}$. Hence, the germs at the origin of $A_*$ and of $\max \{ x^2 + p_1 x : x^2 + p_2 \leq 0 \} = \max \{ -p_2 \pm p_1 \sqrt{-p_2} \} = -p_2 + \sqrt{-p_2} |p_1|$, $p_2 \leq 0$ are $R^+$-equivalent. Adding $p_2$ to this function and, after that, considering new coordinates $\tilde{p}_1 = -p_2$ and $\tilde{p}_2 = p_1$ we obtain singularity 8.

2. If the normal form for $f$ is $-x^2 + p_1 x$ then we have to consider both normal forms for the stationary domain and we obtain singularities $7_\pm$.

**Situation 16.** The origin is a point of type $A_{0,0}^2$

In this situation there are three conditions, namely, $f_x = 0$ and $v_1 = v_2 = 0$ at the origin, for some admissible velocities $v_1$ and $v_2$. As in the previous situations, at the origin the germ of $f$ is $\mathcal{F}^+$-equivalent to the germ of one of the functions $\pm x^2$.

The boundary of the stationary domain is given by $v_1 \cdot v_2 = 0$. By Mather Division Theorem, around the origin equations $v_i = 0$ are equivalent to $x + r_i(p) = 0$, with $1 \leq i \leq 2$, for some smooth functions $r_1$ and $r_2$ vanishing at the origin. Transversality theorems imply that generically the rank of the matrix

$$
\begin{pmatrix}
  r_{1,p_1} & r_{1,p_2} & r_{1,p_3} \\
  r_{2,p_1} & r_{2,p_2} & r_{2,p_3}
\end{pmatrix}
(0)
$$

is maximum. Then, without loss of generality, we can choose $\tilde{p}_1 = r_1(p)$ and $\tilde{p}_2 = r_2(p)$ as new coordinates, preserving the normal form for $f$ and writing the boundary of the
stationary domain as \((x + p_1)(x + p_2) = 0\). Now, considering \(\tilde{x} = x + p_1\), \(\tilde{p}_1 = \mp 2p_1\) and \(\tilde{p}_2 = p_2 - p_1\) we obtain that \(f\) is \(\mathcal{F}^+\)-equivalent to \(\pm x^2 + p_1x\) and that the boundary of the stationary domain is given by \(x(x + p_2) = 0\). By Theorem 2.9 we have to consider the following four normal forms for the stationary domain \(S\): \(\{\pm x(x + p_2) \leq 0\}\) and \(\{x \leq 0 \lor \pm(x + p_2) \leq 0\}\).

1. If the normal form for \(f\) is \(x^2 + p_1x\) then, because \(f(0, 0)\) corresponds to a maximum of \(f(\cdot, 0)\), we just have to consider \(S = \{x(x + p_2) \leq 0\}\). Hence, the germs at the origin of \(A_s\) and of \(\max\{x^2 + p_1x : x(x + p_2) \leq 0\} = \max\{0, p_2^2 - p_1p_2\}\) are \(R^+\)-equivalent. Considering a new coordinate \(\tilde{p}_1 = (p_2 - p_1)/2\) and, after that, adding \(-p_1p_2\) to the resultant function we obtain singularity 9.

2. If the normal form for \(f\) is \(-x^2 + p_1x\) then we have to consider all normal forms for the stationary domain. Considering both “+” cases of \(S\) we obtain singularities 10 and 11 (to obtain this last singularity we consider a new coordinate \(\tilde{p}_2 = -p_2\)). For the “-” cases we have:

(a) \(S = \{x(x + p_2) \geq 0\}\): considering a new coordinate \(\tilde{p}_2 = -(p_1 + p_2)\) we obtain singularity 10 after calculating the maximum.

(b) \(S = \{x \leq 0 \lor x + p_2 \geq 0\}\): considering a new coordinate \(\tilde{p}_2 = p_1 + p_2\) we obtain singularity 11 after calculating the maximum.

**Situation 17.** The origin is a point of type \(A_2^2\)

In this situation there are four conditions, namely, \(f_x = 0\) and \(v = v_x = v_{xx} = 0\) at the origin, for some admissible velocity \(v\). As in the previous situations, at the origin the germ of \(f\) is \(\mathcal{F}^+\)-equivalent to the germ of one of the functions \(\pm x^2\).

The boundary of the stationary domain is given by \(v = 0\). By Mather Division Theorem, around the origin equation \(v = 0\) is equivalent to \(x^3 + r_1(p)x^2 + r_2(p)x + r_3(p) = 0\), for some smooth functions \(r_1, r_2\) and \(r_3\) vanishing at the origin. Transversality theorems imply that generically

\[
\begin{vmatrix}
   r_{3,p_1} & r_{3,p_2} & r_{3,p_3} \\
   r_{2,p_1} & r_{2,p_2} & r_{2,p_3} \\
   r_{1,p_1} & r_{1,p_2} & r_{1,p_3}
\end{vmatrix} (0) \neq 0.
\]

Hence, considering new coordinates \(\tilde{p}_i = r_i(p)\), with \(1 \leq i \leq 3\), the normal form for \(f\) is preserved and the boundary of the stationary domain takes the form \(x^3 + p_1x^2 + p_2x + p_3 = 0\). We can always assume that \(S = \{x^3 + p_1x^2 + p_2x + p_3 \leq 0\}\) because otherwise we consider \(\tilde{x} = -x, \tilde{p}_1 = -p_1, \tilde{p}_3 = -p_3\). Then, because \(f(0, 0)\) corresponds to a maximum of \(f(\cdot, 0)\), the normal form for \(f\) must be \(-x^2\).

So, the germs at the origin of \(A_s\) and of \(\max\{-x^2 : x^3 + p_1x^2 + p_2x + p_3 = 0\}\) are
2.4. Singularities of the optimal averaged profit for stationary strategies

\( R^+ \)-equivalent. Doing

\[
\tilde{x} = x + \frac{1}{3} p_1, \quad \tilde{p}_1 = 2 \frac{2}{3} p_1, \quad \tilde{p}_2 = p_2 - \frac{1}{3} p_1^2 \quad \text{and} \quad \tilde{p}_3 = p_3 + \frac{2}{27} p_1^3 - \frac{1}{3} p_1 p_2
\]

we obtain singularity 19.

**Situation 18.** The origin is a point of type \( A_{0,1}^2 \)

In this situation there are four conditions, namely, \( f_x = 0 \) and \( v_1 = v_2 = v_{2,x} = 0 \) at the origin, for some admissible velocities \( v_1 \) and \( v_2 \). As in the previous situations, at the origin the germ of \( f \) is \( \mathcal{F}^+ \)-equivalent to the germ of one of the functions \( \pm x^2 \).

The boundary of the stationary domain is given by \( v_1 \cdot v_2 = 0 \). By Mather Division Theorem, around the origin equations \( v_1 = 0 \) and \( v_2 = 0 \) are equivalent to \( x + r_1(p) = 0 \) and \( x^2 + r_2(p)x + r_3(p) = 0 \), respectively, for some smooth functions \( r_1, r_2 \) and \( r_3 \) vanishing at the origin. Transversality theorems imply that generically

\[
\begin{vmatrix}
  r_{1,p_1} & r_{1,p_2} & r_{1,p_3} \\
  r_{3,p_1} & r_{3,p_2} & r_{3,p_3} \\
  r_{2,p_1} & r_{2,p_2} & r_{2,p_3}
\end{vmatrix} = 0
\]

Considering new coordinates \( \tilde{p}_i = r_i(p) \), with \( 1 \leq i \leq 3 \), the normal form for \( f \) is preserved and the boundary of the stationary domain takes the form \( (x + p_1)(x^2 + p_2 x + p_3) = 0 \).

Now, considering \( \tilde{x} = x + p_1, \tilde{p}_1 = \mp 2 p_1, \tilde{p}_2 = p_2 - 2 p_1, \tilde{p}_3 = p_3 - p_1 p_2 + p_1^2 \) we obtain that \( f \) is \( \mathcal{F}^+ \)-equivalent to \( \pm x^2 + p_1 x \) and that the boundary of the stationary domain is given by \( x(x^2 + p_2 x + p_3) = 0 \). By Theorem 2.9 we have to consider the following three normal forms for the stationary domain \( S \): \( \{x(x^2 + p_2 x + p_3) \leq 0\}, \{x \leq 0 \lor \pm (x^2 + p_2 x + p_3) \leq 0\} \). Then, because \( f(0,0,0) \) corresponds to a maximum of \( f(\cdot,0) \), the normal form for \( f \) must be \( -x^2 + p_1 x \) and we obtain singularities 20 and 21±.

**Situation 19.** The origin is a point of type \( A_{0,0,0}^2 \)

In this situation there are four conditions, namely, \( f_x = 0 \) and \( v_1 = v_2 = v_3 = 0 \) at the origin, for some admissible velocities \( v_1, v_2 \) and \( v_3 \). As in the previous situations, at the origin the germ of \( f \) is \( \mathcal{F}^+ \)-equivalent to the germ of one of the functions \( \pm x^2 \).

The boundary of the stationary domain is given by \( v_1 \cdot v_2 \cdot v_3 = 0 \). By Mather Division Theorem, around the origin equations \( v_1 = 0 \) are equivalent to \( x + r_1(p) = 0 \), with \( 1 \leq i \leq 3 \), for some smooth functions \( r_1, r_2 \) and \( r_3 \) vanishing at the origin. Transversality theorems imply that generically

\[
\begin{vmatrix}
  r_{1,p_1} & r_{1,p_2} & r_{1,p_3} \\
  r_{2,p_1} & r_{2,p_2} & r_{2,p_3} \\
  r_{3,p_1} & r_{3,p_2} & r_{3,p_3}
\end{vmatrix} = 0
\]
So, considering new coordinates $\tilde{p}_i = r_i(p)$ ($1 \leq i \leq 3$), the normal form for $f$ is preserved and the boundary of the stationary domain takes the form $(x + p_1)(x + p_2)(x + p_3) = 0$. Now, considering $\tilde{x} = x + p_1$, $p_1 = \mp 2p_1$, $p_2 = p_2 - p_1$, $p_3 = p_3 - p_1$ we obtain that $f$ is $\mathcal{F}^+$-equivalent to $\pm x^2 + p_1 x$ and the boundary of the stationary domain is given by $x(x + p_2)(x + p_3) = 0$. By Theorem 2.9 we have to consider the following five normal forms for $S$: $\{ x(x + p_2) \leq 0 \lor x(x + p_3) \leq 0 \}$, $\{ \pm x(x + p_2) \leq 0 \lor x(x + p_3) \geq 0 \}$, $\{ x \leq 0 \lor x + p_2 \leq 0 \lor \pm(x + p_3) \leq 0 \}$.

1. If the normal form for $f$ is $x^2 + p_1 x$ then, because $f(0,0)$ corresponds to a maximum of $f(\cdot,0)$, we just have to consider the first normal form for $S$. The germs at the origin of $A_s$ and of $\max \{ x^2 + p_1 x : x(x + p_2) \leq 0 \lor x(x + p_3) \leq 0 \} = \max \{ 0, p_2^2 - p_1 p_2, p_3^2 - p_1 p_3 \}$ are $\mathcal{F}^+$-equivalent. Adding $\frac{p_1^2}{4}$ to this function and then considering new coordinates $\tilde{p}_1 = \frac{p_1}{2}$, $\tilde{p}_2 = p_2 - \frac{p_1}{2}$ and $\tilde{p}_3 = p_3 - \frac{p_1}{2}$ we obtain singularity 22.

2. If the normal form for $f$ is $-x^2 + p_1 x$ then we have to consider all normal forms for the stationary domain. With the first normal form for $S$ we obtain singularity 23. For $S = \{ x \leq 0 \lor x + p_2 \leq 0 \lor x + p_3 \leq 0 \}$ we obtain singularity 24 after considering $\tilde{p}_2 = -p_2$ and $\tilde{p}_3 = -p_3$. For the “-” cases we have:

(a) $S = \{ x(x + p_2) \leq 0 \lor x(x + p_3) \geq 0 \}$: considering a new coordinate $\tilde{p}_3 = -(p_1 + p_3)$ we obtain singularity 23 after calculating the maximum.

(b) $S = \{ x(x + p_2) \geq 0 \lor x(x + p_3) \geq 0 \}$: considering new coordinates $\tilde{p}_2 = -(p_1 + p_2)$ and $\tilde{p}_3 = -(p_1 + p_3)$ we obtain singularity 23 after calculating the maximum.

(c) $S = \{ x \leq 0 \lor x + p_2 \leq 0 \lor x + p_3 \geq 0 \}$: considering new coordinates $\tilde{p}_2 = -p_2$ and $\tilde{p}_3 = p_1 + p_3$ we obtain singularity 24 after calculating the maximum.

**Situation 20.** The origin is a point of type $A_0^3$

In this situation there are three conditions, namely, $f_x = f_{xx} = 0$ and $v = 0$ at the origin, for some admissible velocity $v$. By classical results on Singularity Theory [6] we know that generically at the origin the germ of $f$ is $\mathcal{F}^+$-equivalent to the germ of the function $x^3 + \alpha(p)x$, where $\alpha$ is a smooth function with $\alpha(0) = 0$.

The boundary of the stationary domain is given by $v = 0$. By Mather Division Theorem, around the origin equation $v = 0$ is equivalent to $x + r(p) = 0$, for some smooth function $r$ vanishing at the origin. Transversality theorems imply that generically the matrix

$$
\begin{pmatrix}
0 & \alpha_{p_1} & \alpha_{p_2} & \alpha_{p_3} \\
6 & 0 & 0 & 0 \\
1 & r_{p_1} & r_{p_2} & r_{p_3}
\end{pmatrix}
$$

has maximum rank. Then, generically we can suppose that $\alpha_{p_1}$ and $r_{p_2}$ do not vanish at the origin. Considering new coordinates $\tilde{p}_1 = \alpha(p)$ and $\tilde{p}_2 = -r(p)$, the family $f$ and
the boundary of the stationary domain take the form $x^3 + p_1 x$, up to $\mathcal{F}^+$-equivalence, and $x = p_2$, respectively. Then because $f(0,0)$ corresponds to a maximum of $f(\cdot,0)$, $S = \{ x \leq p_2 \}$ and so, the germs at the origin of $A_s$ and of $\max \{ x^3 + p_1 x : x \leq p_2 \}$ are $R^+$-equivalent. Doing $\tilde{x} = x - p_2$, $\tilde{p}_1 = 3p_2$ and $\tilde{p}_2 = p_1 + 3p_2^2$ we conclude that the germs at the origin of $A_s$ and of $\max \{ x^3 + p_1 x^2 + p_2 x : x \leq 0 \}$ are $R^+$-equivalent. We obtain singularity 6.

**Situation 21.** The origin is a point of type $A_{1,3}^3$

In this situation there are four conditions, namely, $f_x = f_{xx} = 0$ and $v = v_x = 0$ at the origin, for some admissible velocity $v$. As in the previous situation, at the origin the germ of $f$ is $R^+$-equivalent to the germ of the function $x^3 + \alpha(p)x$.

The boundary of the stationary domain is given by $v = 0$. By Mather Division Theorem, around the origin equation $v = 0$ is equivalent to $x^2 + r_1(p)x + r_2(p) = 0$, for some smooth functions $r_1$ and $r_2$ vanishing at the origin. Transversality theorems imply that generically

$$\begin{vmatrix}
\alpha_{p_1} & \alpha_{p_2} & \alpha_{p_3} \\
r_2,p_1 & r_2,p_2 & r_2,p_3 \\
r_1,p_1 & r_1,p_2 & r_1,p_3
\end{vmatrix} (0) \neq 0.$$ 

Hence, considering new coordinates $\tilde{p}_1 = \alpha(p)$, $\tilde{p}_2 = r_1(p)$ and $\tilde{p}_3 = r_2(p)$, the family $f$ and the boundary of the stationary domain take the form $x^3 + p_1 x$, up to $\mathcal{F}^+$-equivalence, and $x^2 + p_2 x + p_3 = 0$, respectively. Now, doing $\tilde{x} = x + \frac{p_2}{2}$, $\tilde{p}_1 = -\frac{3}{2}p_2$, $\tilde{p}_2 = p_1 + \frac{3}{4}p_2^2$ and $\tilde{p}_3 = p_3 - \frac{p_2^2}{4}$ we obtain that $f$ and the boundary of the stationary domain take the form $x^3 + p_1 x^2 + p_2 x$, up to $\mathcal{F}^+$-equivalence, and $x^2 + p_2 = 0$, respectively. By Theorem 2.9 we have to consider the following two normal forms for the stationary domain $S$: $\{ \pm (x^2 + p_3) \leq 0 \}$. However, because $f(0,0)$ corresponds to a maximum of $f(\cdot,0)$, we just have to consider the “+” case for $S$ and so we get singularity 16.

**Situation 22.** The origin is a point of type $A_{0,0,3}$

In this situation there are four conditions, namely, $f_x = f_{xx} = 0$ and $v_1 = v_2 = 0$ at the origin, for some admissible velocities $v_1$ and $v_2$. As in the previous situations, at the origin the germ of $f$ is $\mathcal{F}^+$-equivalent to the germ of the function $x^3 + \alpha(p)x$.

The boundary of the stationary domain is given by $v_1 \cdot v_2 = 0$. By Mather Division Theorem, around the origin equation $v_i = 0$ is equivalent to $x + r_i(p) = 0$, with $1 \leq i \leq 2$, for some smooth functions $r_1$ and $r_2$ vanishing at the origin. As in the previous situation, due to transversality theorems we consider new coordinates $\tilde{p}_1 = \alpha(p)$, $\tilde{p}_2 = r_1(p)$ and $\tilde{p}_3 = r_2(p)$, obtaining that the family $f$ and the boundary of the stationary domain take the form $x^3 + p_1 x$, up to $\mathcal{F}^+$-equivalence, and $(x + p_2)(x + p_3) = 0$, respectively. Now, doing $\tilde{x} = x + p_2$, $\tilde{p}_1 = -3p_2$, $\tilde{p}_2 = p_1 + 3p_2^2$ and $\tilde{p}_3 = p_3 - p_2$ we obtain that $f$ and the
boundary of the stationary domain take the form $x^3 + p_1 x^2 + p_2 x$, up to $\mathcal{F}^+$-equivalence, and $x(x + p_3) = 0$, respectively. By Theorem 2.9 we have to consider the following four normal forms for the stationary domain $S$: $\{ \pm x(x + p_3) \leq 0 \}$ and $\{ x \leq 0 \lor \pm (x + p_3) \leq 0 \}$. However, because $f(0, 0)$ corresponds to a maximum of $f(\cdot, 0)$, we just have to consider the “+” cases for $S$ and so we get singularities 17 and 18 (to obtain this last normal form we consider $\tilde{p}_3 = -p_3$).

**Situation 23.** The origin is a type $A_0^4$

In this situation there are four conditions, namely, $f_x = f_{xx} = f_{xxx} = 0$ and $v = 0$ at the origin, for some admissible velocity $v$. By classical results on Singularity Theory [6] we know that generically the germ of $f$ at the origin is $\mathcal{F}^+$-equivalent to the germ of one of the functions $\pm x^4 + \alpha(p)x^2 + \beta(p)x$, where $\alpha$ and $\beta$ are smooth functions with $\alpha(0) = 0 = \beta(0)$.

The boundary of the stationary domain is given by $v = 0$. By Mather Division Theorem, around the origin equation $v = 0$ is equivalent to $x + r(p) = 0$, for some smooth function $r$ vanishing at the origin. Due to transversality theorems we consider new coordinates $\tilde{p}_1 = \alpha(p)$, $\tilde{p}_2 = \beta(p)$ and $\tilde{p}_3 = r(p)$, obtaining that the family $f$ and the boundary of the stationary domain take the form $\pm x^4 + p_1 x^2 + p_2 x$, up to $\mathcal{F}^+$-equivalence, and $x + p_3 = 0$, respectively. We can always assume that $S = \{ x + p_3 \leq 0 \}$ (otherwise we consider $\tilde{x} = -x$, $\tilde{p}_3 = -p_3$ and $\tilde{p}_2 = -p_2$). Then, because $f(0, 0)$ corresponds to a maximum of $f(\cdot, 0)$, the normal formal for $f$ must be $-x^4 + p_1 x^2 + p_2 x$. Considering $\tilde{x} = x + p_3$, $\tilde{p}_1 = 4p_3$, $\tilde{p}_2 = p_1 - 6p_3^2$ and $\tilde{p}_3 = p_2 - 2p_1 p_3 + 4p_3^3$ we obtain singularity 15.

### 2.4.2 Competition singularities

In this subsection we classify all generic singularities of the optimal averaged profit for stationary strategies when there is competition between them, that is, when for a given parameter value $p$, the closure of $S^*$ has more than one point of the form $(x, p)$, which are said said to be competing to provide the profit $A_s(p)$ (see Definition 2.10 presented in the beginning of this section).

**Lemma 2.15** Consider a $k$-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional compact manifold, $k \leq 3$. Generically, if there are two points competing to provide the optimal profit $A_s$ having different levels,* then the point having the highest level provides:

*Two points have same level if the family of profit densities has the same value at these points.
2.4. Singularities of the optimal averaged profit for stationary strategies

- singularity 3 of Table 2.4, if \( k = 1 \);
- one of singularities 3, 7, and 8 of Tables 2.4 and 2.5, if \( k = 2 \);
- one of singularities 3, 7, 8, 16, 25, and 27 of Tables 2.4-2.6, if \( k = 3 \).

**Proof**: All singularities listed in this lemma have in common the property that any neighborhood of the origin contains values of the parameter for which they are not defined. All remaining singularities from Tables 2.4-2.6 are continuous at the origin and are well defined in a neighborhood of it. Suppose \((x_1, p)\) and \((x_2, p)\) are two points competing to provide the optimal profit \( A_s(p) \). We select \( p \) as the origin. Let \((x_2, 0)\) be the point having the highest level, that is \( f(x_1, 0) < f(x_2, 0) \). It is clear that near the origin \( A_s(p) = \max\{A_1^s(p), A_2^s(p)\} \), where \( A_i^s(p) \) is the maximum of \( f \) restricted to a sufficiently small neighborhood \( U_i \) of \((x_i, 0)\) taken over \( x \in S(p) \), that is,

\[
A_i^s(p) = \max_{x \in S(p)} f|_{U_i}(x, p), \quad 1 \leq i \leq 2.
\]

Using Theorem 2.14 we obtain the possible generic singularities for each \( A_i^s \). So, if \((x_2, 0)\) provides a singularity not stated in this lemma, \( A_2^s \) is continuous at the origin and is defined for all values in a neighborhood of it. Then, since \( A_2^s(0) > A_1^s(0) \) and \( A_1^s \) is upper semicontinuous, we conclude that the germ of \( A_s \) at the origin coincides with \( A_2^s \). But this contradicts the fact that there are two points competing to provide the optimal profit.

**Remark.** Note that the continuity of singularities at the origin plays an important role in the proof of this result.

**Lemma 2.16** Consider a \( k \)-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional compact manifold \( M, k \leq 3 \). Suppose that there are exactly \( N \) distinct points \( Q_i, i = 1, ..., N \), competing to provide the optimal profit with exactly \( l \) distinct levels, \( 1 \leq l \leq N \), such that each point \( Q_i \) provides a codimension \( c_i \) singularity from Tables 2.4-2.6. Then, generically,

\[
\sum_{i=1}^{N} c_i + N - l \leq k.
\]

**Proof**: Consider a point \((Q_1, \cdots, Q_N) \in (M \times P)^{(N)}\) such that:

- each \( Q_i \) provides a codimension \( c_i \) singularity from Tables 2.4-2.6,
- \( \pi(Q_1) = \cdots = \pi(Q_N) \), where \( \pi \) is the projection on the parameter space,
- all points together provide exactly \( l \) distinct levels.
At such a point, a family of pairs of polydynamical systems and profit densities satisfies exactly $\mu$ independent equalities (among other conditions) with

$$\mu = \sum_{i=1}^{N} (c_i + 1) + (N - 1)k + N - l = \sum_{i=1}^{N} c_i + N(k + 2) - l - k.$$ 

Consider the subset $Y$ of $(M \times P)^{(N)}$ consisting of the points at which just these equalities are satisfied. In the multijet bundle $J_N^5 (M \times P, \mathbb{R})$, the set $W$ of multijets of families at points of $Y$ is a codimension $\mu$ closed submanifold. Due to Multijet Transversality Theorem and the compactness of the phase space $M$, for a generic family $(V, f)$, the multijet extension $j_N^5 (V, f)$ is transversal to $W$ and, therefore, $(j_N^5 (V, f))^{-1}(W)$ is either empty or is a codimension $\mu$ submanifold of $(M \times P)^{(N)}$. But $(j_N^5 (V, f))^{-1}(W) = Y$ and $Y$ is a nonempty set. Therefore, generically its codimension can not be greater than the dimension $N(k + 1)$ of $(M \times P)^{(N)}$, that is, $\mu \leq N(k + 1)$, and so, $\sum_{i=1}^{N} c_i + N - l \leq k$.

**Corollary 2.17:** Consider a $k$-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional compact manifold, $k \leq 3$. Generically, there can not exist more than $k + 1$ points competing to provide the optimal averaged profit for stationary strategies.

**Proof:** Generically points with the lowest level can provide all the singularities of Tables 2.4-2.6. Points with other levels can provide only the singularities listed on Lemma 2.15. Therefore, the lowest possible codimensions are 0 for the lowest level and 1 for all the other levels. Hence, the sum of all codimensions is always not less than $l - 1$ and so, by the previous lemma, in a generic case $N \leq k + l - \sum_{i=1}^{n} c_i \leq k + 1$. 

In order to describe the various generic situations of competition we use the notation

$$C(T_1, \cdots, T_N)$$

To denote the competition of exactly $N$ points where:

1. $T_1, \ldots, T_N$ are the types of such points,
2. for all $i < N$, the level of the point of type $T_i$ is not higher than the level of the point of type $T_{i+1}$,
3. distinct levels are marked replacing commas by semicolons.

For example, $C(A_0^1, A_0^2; A_1^1)$ denotes the competition of points of type $A_0^1$, $A_0^2$ and $A_1^1$ with $A_0^1$ and $A_0^2$ having the same level and $A_1^1$ having a higher one.
2.4. Singularities of the optimal averaged profit for stationary strategies

**Theorem 2.18:** For a generic \( k \)-parameter family of pairs of polydynamical systems and profit densities on a 1-dimensional compact manifold, the germ of the optimal profit for stationary strategies at a parameter value with competition is, up to the equivalence pointed out in the fourth column, the germ at the origin of one of the functions from the second column of:

- **Table 2.7**, if \( k = 1 \),
- **Tables 2.7 and 2.8**, if \( k = 2 \),
- **Tables 2.7, 2.8 and 2.9**, if \( k = 3 \).

In Column 3 we can always replace \( A_{01} \) by \( I^2 \).

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Situation</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(</td>
<td>p_1</td>
<td>)</td>
</tr>
<tr>
<td>2</td>
<td>( \max{0, \sqrt{p_1} + 1} )</td>
<td>( C(A_{01}; A_{11}) )</td>
<td>( \Gamma )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Situation</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( \max{p_1</td>
<td>p_1</td>
<td>, p_2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \max{\sqrt{p_1}, p_2} )</td>
<td>( C(A_{01}^1, A_{11}^1) )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \max{</td>
<td>p_1</td>
<td>, p_2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \max{p_1</td>
<td>p_1</td>
<td>, \sqrt{p_2} + 1} )</td>
</tr>
<tr>
<td>7</td>
<td>( \max{\sqrt{p_1}, \sqrt{p_2} + 1} )</td>
<td>( C(A_{11}^1; A_{11}^1) )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>( \max{</td>
<td>p_1</td>
<td>, \sqrt{p_2} + 1} )</td>
</tr>
<tr>
<td>9</td>
<td>( \max{0, \sqrt{p_1}</td>
<td>p_2</td>
<td>+ 1} )</td>
</tr>
<tr>
<td>10</td>
<td>( \max{0, -x^2 + p_1 x + 1 : x^2 + p_2 \leq 0} )</td>
<td>( C(A_{01}^1; A_{11}^1; A_{11}^1) )</td>
<td></td>
</tr>
</tbody>
</table>

**Remark.** Comparing these singularities with those presented in the previous section on Theorem 2.14, we see that there are common singularities. In particular, the singularity obtained with the competition of points of type \( A_{i_1}^1, \ldots, A_{i_N}^1 \), having the same level, coincides with the singularity presented on Theorem 2.14 obtained when the profit is attained at a point of type \( A_{i_1,\ldots,i_N}^1 \).
Proof: Let \( p_0 \) be a value with competition and let \((x_1, p_0), \ldots, (x_N, p_0)\), \( N \geq 2 \), be the points competing to provide the optimal profit \( A_\ast(p_0) \). We select \( p_0 \) as the origin. As a generalization of the process used in the proof of Lemma 2.15, it is clear that near the origin \( A_\ast(p) = \max_{1 \leq i \leq N} A_i^\ast(p) \), where \( A_i^\ast(p) \) is the maximum of \( f \) restricted to a sufficiently small neighborhood \( U_i \) of \((x_i, 0)\) taken over \( x \in S(p) \). Using Theorem 2.14 we obtain all possible generic singularities for each \( A_i^\ast \).

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Situation</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
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<td>( R^+ )</td>
</tr>
<tr>
<td>13</td>
<td>( \max{x^3 + p_1 x^2 + p_2 x : x \leq 0, p_3} )</td>
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<td></td>
</tr>
<tr>
<td>14</td>
<td>( \max{\sqrt{p_1}</td>
<td>p_2, p_3} )</td>
<td>( C(A_0^1, A_1^2) )</td>
</tr>
<tr>
<td>15±</td>
<td>( \max{-x^2 + p_1 x : \pm(x + p_2) \leq 0, p_3} )</td>
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<td></td>
</tr>
<tr>
<td>16</td>
<td>( \max{p_1 p_2, p_3} )</td>
<td>( C(A_0^1, A_0, 0^2) )</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>( \max{-x^2 + p_1 x : x(x + p_2) \leq 0, p_3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>( \max{-x^2 + p_1 x : x \leq \max(0, p_2); p_3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>( \max{x : x(x^3 + p_1 x^2 + p_2 x + p_3) = 0} )</td>
<td>( C(A_0^1, A_2^1) )</td>
<td></td>
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<tr>
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<td>, \sqrt{p_2 + p_3}} )</td>
</tr>
<tr>
<td>21</td>
<td>( \max{</td>
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<td>,</td>
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<tr>
<td>22</td>
<td>( \max{p_1</td>
<td>p_1</td>
<td>, p_2</td>
</tr>
<tr>
<td>23</td>
<td>( \max{p_1</td>
<td>p_1</td>
<td>, \sqrt{p_2 + p_3}} )</td>
</tr>
<tr>
<td>24</td>
<td>( \max{p_1</td>
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<td>, p_2</td>
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<tr>
<td>25</td>
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<td>, \sqrt{p_2 + p_3}} )</td>
</tr>
<tr>
<td>26</td>
<td>( \max{-x^4 + p_1 x^2 + p_2 x : x \in \mathbb{R}, \sqrt{p_3 + 1}} )</td>
<td>( C(I_4^1; A_1^1) )</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>( \max{x^3 + p_1 x^2 + p_2 x : x \leq 0, \sqrt{p_3 + 1}} )</td>
<td>( C(A_0^3; A_{1,0}^1) )</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>( \max{\sqrt{p_1}</td>
<td>p_2, \sqrt{p_3 + 1}} )</td>
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<tr>
<td>29±</td>
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<td></td>
</tr>
<tr>
<td>30</td>
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<td>( C(A_1^2; A_{1,0}^1) )</td>
<td></td>
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<tr>
<td>31</td>
<td>( \max{-x^2 + p_1 x : x(x + p_2) \leq 0, \sqrt{p_3 + 1}} )</td>
<td>( C(A_0,0^2; A_{1,0}^1) )</td>
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<tr>
<td>32</td>
<td>( \max{-x^2 + p_1 x : x \leq \max(0, p_2); \sqrt{p_3 + 1}} )</td>
<td>( C(A_0^2; A_{1,0}^1) )</td>
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<td>33</td>
<td>( \max{x : x^3 + p_1 x + p_2 = 0, \sqrt{p_3 + 1}} )</td>
<td>( C(A_2^1; A_{1,0}^1) )</td>
<td></td>
</tr>
</tbody>
</table>
Now, using Lemmas 2.15 and 2.16 we conclude which situations of competition must be considered in a generic case. For example, for $k = 3$ and $N = 4$, by Lemma 2.16 we conclude that $\sum_{l=1}^{4} c_l + 4 - l \leq 3$. So,

1. if $l = 1$, then $c_1 = \cdots = c_4 = 0$, and we get situation $C(A_0^1, A_0^1, A_0^1, A_0^1)$.

2. if $l = 2$, then $\sum_{i=1}^{4} c_i \leq 1$, and by Lemma 2.15 we conclude that generically only situation $C(A_0^1, A_0^1, A_0^1; A_1^1)$ can occur.

3. if $l = 3$, then $\sum_{i=1}^{4} c_i \leq 2$ and by Lemma 2.15, we conclude that generically only situation $C(A_0^1, A_0^1; A_1^1; A_1^1)$ can occur.

4. if $l = 4$, generically we also obtain a unique situation: $C(A_0^1; A_1^1; A_1^1; A_1^1)$.

<table>
<thead>
<tr>
<th>N.</th>
<th>Singularities</th>
<th>Situation</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>$\max{\sqrt{p_1}, p_2, \sqrt{p_3} + 1}$</td>
<td>$C(A_0^1, A_1^1)$, $C(A_0^1, A_1^1; A_1^1)$</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>$\max{</td>
<td>p_1</td>
<td>, p_2, \sqrt{p_3} + 1}$</td>
</tr>
<tr>
<td>36</td>
<td>$\max{</td>
<td>p_1</td>
<td>, p_2, \sqrt{p_3} + 1}$</td>
</tr>
<tr>
<td>37</td>
<td>$\max{</td>
<td>p_1</td>
<td>, p_2, \sqrt{p_3} + 1}$</td>
</tr>
<tr>
<td>38</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2}</td>
</tr>
<tr>
<td>39</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2}</td>
</tr>
<tr>
<td>40</td>
<td>$\max{</td>
<td>p_1</td>
<td>, -x^2 + p_2 x + 1 : x^2 + p_3 \leq 0}$</td>
</tr>
<tr>
<td>41</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2}</td>
</tr>
<tr>
<td>42</td>
<td>$\max{</td>
<td>p_1</td>
<td>, -x^2 + p_2 x + 1 : x^2 + p_3 \leq 0}$</td>
</tr>
<tr>
<td>43</td>
<td>$\max{x^3 + p_1 x^2 + p_2 x + 1 : x^2 + p_3 \leq 0}$</td>
<td>$C(A_0^1; A_1^3)$</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>$\max{x, x + 1 : x^4 + p_1 x^2 + p_2 x + p_3 = 0}$</td>
<td>$C(A_0^1; A_3^1)$</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>$\max{0, \sqrt{p_1} + 1, \sqrt{p_2} + p_3 + 1}$</td>
<td>$C(A_0^1, A_1^1)$, $C(A_0^1, A_1^1, A_1^1)$</td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2} + 1, \sqrt{p_3} + 2}$</td>
</tr>
<tr>
<td>47</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2} + 1, \sqrt{p_3} + 2}$</td>
</tr>
<tr>
<td>48</td>
<td>$\max{</td>
<td>p_1</td>
<td>, \sqrt{p_2} + 1, \sqrt{p_3} + 2}$</td>
</tr>
<tr>
<td>49</td>
<td>$\max{0, \sqrt{p_1} + 1, \sqrt{p_2} + 2}$</td>
<td>$C(A_0^1, A_1^2)$</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>$\max{0, -x^2 + p_1 x + 1 : x^2 + p_2 \leq 0, \sqrt{p_3} + 2}$</td>
<td>$C(A_0^1, A_1^2)$</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>$\max{0, \sqrt{p_1} + 1, \sqrt{p_2} + 2}$</td>
<td>$C(A_0^1, A_1^2)$</td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>$\max{0, -x^2 + p_2 x + 2 : x^2 + p_3 \leq 0}$</td>
<td>$C(A_0^1, A_1^2)$</td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>$\max{0, \sqrt{p_1} + 1, \sqrt{p_2} + 2, \sqrt{p_3} + 3}$</td>
<td>$C(A_0^1, A_1^2)$</td>
<td></td>
</tr>
</tbody>
</table>
For each possible situation the idea to complete the proof is the same. By Multijet Transversality Theorem we conclude that generically it is possible, using local changes of coordinates of the form \((x, p) \mapsto (\varphi_1(x, p), h(p))\) around each \((x_i, 0)\), to put simultaneously all germs \(A_s^i\) in a preliminary normal form, namely: \(A_s^i = g_i + \varphi_1\), where \(g_i\) has one of the normal forms presented in Tables 2.4-2.6, such that if \(i \neq j\) then \(g_i\) and \(g_j\) depend on different components \(p_1, p_2, \ldots, p_k\) of \(p\), and \(\varphi_i\) are smooth functions of the parameter. So the optimal profit is \(R^+\)-equivalent to: \(A_s = \max\{g_1, g_2 + \gamma_1, \ldots, g_N + \gamma_N\}\), where \(\gamma_i = \varphi_i - \varphi_1\). Using again Multijet Transversality Theorem we are now able to put \(A_s\) in one of the normal forms of Tables 2.7-2.9 through \(R^+\)-equivalence if all the points in competition have the same level, and through \(\Gamma\)-equivalence if the points in competition have at least two distinct levels. To make things clear we present two cases, namely \(C(I^2, A_0^2)\) and \(C(A_0^2; A_1^1)\) for \(k = 2\).

1. \(C(I^2, A_0^2)\)

Suppose that \((x_1, 0)\) and \((x_2, 0)\) are points of type \(I^2\) and \(A_0^2\), respectively, having the same level and competing to provide the optimal profit \(A_s(0)\). In this case there are four conditions, namely, \(f_x(x_1, 0) = 0\), \(v(x_2, 0) = 0\), \(f_x(x_2, 0) = 0\) and \(f(x_1, 0) - f(x_2, 0) = 0\). Then, as it was seen in situation 14 of the previous section, we can consider a local coordinate \((y, p)\) with origin at \((x_2, 0)\) where \(f(y, p) = -y^2 + \phi(p)\) and \(S = \{y \leq p_1\}\), for some smooth function \(\phi\). By convenience, we select \(p_1 = 1/\sqrt{2p_1}\) and so, \(S = \{y \leq \sqrt{2p_1}\}\) and \(v(y, p) = (y - \sqrt{2p_1}) \cdot V(y, p)\), for some smooth function \(V\) that does not vanish at the origin. Once \(f_x(x_1, 0) = 0 \neq f_x(x_1, 0)\), the Implicit Function Theorem implies the existence of a smooth function \(\gamma\) of the parameter such that \(\gamma(0) = x_1\) and \(f_x(\gamma(p), p) = 0\) for every \(p\) close to the origin. Therefore, around the origin, \(A_s\) is \(R^+\)-equivalent to

\[
\max \left\{ \max_{y \leq \sqrt{2p_1}} -y^2 + \phi(p); f(\gamma(p), p) \right\},
\]

which is clearly \(R^+\)-equivalent to the function \(\max\{0, p_1|p_1| - p_1^2 + \phi(p) - f(\gamma(p), p)\}\).

Transversality theorems imply that generically the matrix

\[
\begin{pmatrix}
    f_{xx}(x_1, 0) & 0 & f_{xp_1}(x_1, 0) & f_{xp_2}(x_1, 0) \\
    0 & f_{yy}(x_2, 0) & f_{yp_1}(x_2, 0) & f_{yp_2}(x_2, 0) \\
    0 & v_{yp}(x_2, 0) & v_{p_1}(x_2, 0) & v_{p_2}(x_2, 0) \\
    f_{x}(x_1, 0) & -f_y(x_2, 0) & f_{p_1}(x_1, 0) - f_{p_1}(x_2, 0) & f_{p_2}(x_1, 0) - f_{p_2}(x_2, 0)
\end{pmatrix}
\]

has maximal rank. But \(f_x(x_1, 0), f_{yp_1}(x_2, 0), f_{yp_2}(x_2, 0)\) and \(v_{p_1}(x_2, 0)\) vanish and so, generically \(f_{p_2}(x_1, 0) - \phi_{p_2}(0) \neq 0\). Hence, in a generic case we have

\[
\frac{\partial}{\partial p_2} \left( p_1^2 - \phi(p) + f(\gamma(p), p) \right)(0) = f_{p_2}(x_1, 0) - \phi_{p_2}(0) \neq 0,
\]
and then we can consider a new coordinate $\tilde{p}_2 = p_1^2 - \phi(p) + f(\gamma(p), p)$ to get singularity 3 of Table 2.8.

2. $C(A_0^2; A_1^1)$

Suppose that $(x_1, 0)$ and $(x_2, 0)$ are points of type $A_0^2$ and $A_1^1$, respectively, having distinct levels and competing to provide the optimal profit $A_s(0)$. In this case there are four conditions, namely, $f_x(x_1, 0) = 0$, $v_1(x_1, 0) = 0$ and $v_2(x_2, 0) = v_{2,x}(x_2, 0) = 0$, for some admissible velocities $v_1$ and $v_2$. As before, we can consider a local coordinate $(y, p)$ with origin at $(x_1, 0)$ where $f(y, p) = y^2 + \phi(p)$ and $S = \{ y \leq \sqrt{2}p_1 \}$, for some smooth function $\phi$. Then, $v_1(y, p) = (y - \sqrt{2}p_1) \cdot V_1(y, p)$ for some smooth function $V_1$ that does not vanish at the origin.

Since $v_2(x_2, 0) = v_{2,x}(x_2, 0) = 0 \neq v_{2,xx}(x_2, 0)$ and $f_x(x_2, 0) \neq 0$, classical results on Singularity Theory [6] and Mather Division Theorem imply that there exists a local coordinate $z$ with origin at $x_2$ depending smoothly on $p$ where $f(z, p) = z + \varphi(p)$ and $v(z, p) = (z^2 - a(p))V_2(z, p)$, for some smooth functions $\varphi$, $a$ and $V_2$ with $a(0) = 0$ and $V_2(x_2, 0) \neq 0$ (the procedure is analogous to the one that is used in the previous section).

Transversality theorems imply that generically the matrix

$$
\begin{pmatrix}
  f_{yy}(x_1, 0) & 0 & f_{yyp_1}(x_1, 0) & f_{yyp_2}(x_1, 0) \\
  v_1,y(x_1, 0) & 0 & v_{1,y}\gamma_1(x_1, 0) & v_{1,y}\gamma_2(x_1, 0) \\
  0 & v_{2,z}(x_2, 0) & v_{2,y}\gamma_1(x_2, 0) & v_{2,y}\gamma_2(x_2, 0) \\
  0 & v_{2,z}(x_2, 0) & v_{2,z}\gamma_1(x_2, 0) & v_{2,z}\gamma_2(x_2, 0)
\end{pmatrix}
$$

has maximal rank. But because $f_{yyp_1}(x_1, 0)$, $f_{yyp_2}(x_1, 0)$, $v_{1}\gamma_1(x_1, 0)$ and $v_{2}\gamma_2(x_2, 0)$ also vanish, we get that generically $v_{2,y}\gamma_1(x_2, 0) \neq 0$ and so $a_{p_2}(0) \neq 0$. Consequently, choosing a new coordinate $\tilde{p}_2 = a(p)$ we write the boundary of the stationary domain around $(x_2, 0)$ as $z^2 = p_2$.

Note that after this coordinate change, the normal forms for $f$ around $(x_1, 0)$ and $(x_2, 0)$ are the previous ones but with different functions $\phi$ and $\varphi$. Therefore, we conclude that around the origin, $A_s$ is $R^+$-equivalent to the function

$$
\max \left\{ \max_{y \leq \sqrt{2}p_1} -y^2 + \phi(p); \max_{z^2 \leq p_2} z + \varphi(p) \right\},
$$

where $\phi(0) < \varphi(0)$. This function is obviously $R^+$-equivalent to the function

$$
\max \left\{ \max_{y \leq \sqrt{2}p_1} -y^2; \max_{z^2 \leq p_2} z + \varphi(p) \right\} = \max \left\{ \max \{-2p_1^2; 0, p_1 \geq 0\}; \sqrt{p_2} + \varphi(p) \right\}
$$

for a new smooth function $\varphi$ positive at the origin. Now, this function is $R^+$-equivalent to

$$
\max \left\{ p_1|p_1|; \sqrt{p_2} + p_1^2 + \varphi(p) \right\}.
$$
Define $\psi(p) = p_1^2 - \phi(p) + \phi_1(p_2, p)$ and note that $\psi(0) > 0$. The change of coordinates 
$(p, a) \mapsto \left( \frac{p_1}{\sqrt{\psi(p)}}, \frac{p_2}{\psi_2(p)}, \frac{a}{\psi_1(p)} \right)$ leads to singularity 6 of Table 2.8 ($a$ is a coordinate in the real axis, where the family of profit densities takes values).
Chapter 3

Level cycles

In this chapter we present the classification of all generic singularities of the optimal averaged profit when it is provided by level cycles.

In Section 1.1 we introduced basic concepts concerning cyclic strategies and, to simplify, we considered the case without parameter. Now we consider the parameter and so,

1. the velocity $v_c$ that is used in a $c$-level cycle is $v_c(x,p) = \begin{cases} v_{\text{min}}(x,p), & f(x,p) > c \\ v_{\text{max}}(x,p), & f(x,p) \leq c \end{cases}$

2. the period $T$, profit $P$ and averaged profit $A$ along level cycles are functions defined on the product space of the parameter space by the real line by

$$T(p,c) = \int \frac{1}{v_c(x,p)} \, dx, \quad P(p,c) = \int \frac{f(x)}{v_c(x,p)} \, dx \quad \text{and} \quad A(p,c) = \frac{P(p,c)}{T(p,c)}.$$ 

In the following theorem we joint some results presented in [9] and [12]. It plays an important role along this chapter and it is the base of the process to obtain now the normal forms for the optimal averaged profit.

**Theorem 3.1 ([9])**: When, for a fixed value $p_0$ of the parameter, the maximum averaged profit is provided by a $c_0$-level cycle then:

1. the period $T$, profit $P$ and averaged profit $A$ along level cycles are continuous functions,

2. near $c_0$ the averaged profit $A$ along a level cycle is a differentiable function of the level,

3. near $p_0$ the optimal profit is the unique solution $c = c(p)$ of equation $c = A(p,c)$ such that $c(p_0) = c_0$. 

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4. if \( A \) is a \( C^k \)-function around \((p_0, c_0)\) then the optimal profit is a \( C^k \)-function around \( p_0 \), if the differentiable profit density has a finite number of critical points and the maximum and minimum velocities of the continuous control system are equal at isolated points only.

**Definition 3.2:** Consider a family of pairs of polydynamical systems and profit densities on the circle. For each parameter value whose optimal averaged profit is provided by a level cycle we define the optimal averaged profit \( A_l \) for level cycles

\[ A_l(p) = \max_c A(p, c), \]

where \( A \) is the averaged profit along level cycles.

**Lemma 3.3:** Consider a family of polydynamical systems on the circle. Families of profit densities that coincide up to the sum with a smooth function of the parameter provide the same optimal averaged profit for level cycles up to \( R^+-\)equivalence.

**Proof:** Let \( f_1 \) and \( f_2 \) be families of profit densities such that \( f_2(x, p) = f_1(x, p) - F(p) \). Let \( A_l^f(p, c) \) and \( A_l^{f_i}(p) \) denote the averaged profit and the optimal averaged profit for level cycles corresponding to each \( f_i \). It is easy to see that \( A_l^f(p, c) = A_l^{f_1}(p, c + F(p)) - F(p) \). In fact,

\[
A_l^f(p, c) = \max_c A(p, c, c + F(p)) - F(p).
\]

Then,

\[
A_l^f(p, c) = \max_c A_l^{f_1}(p, c + F(p)) - F(p) = A_l^{f_1}(p) - F(p).
\]

and so,

\[
A_l^f(p) = \max_c A_l^f(p, c) = \max_c A_l^{f_1}(p, c + F(p)) - F(p) = A_l^{f_1}(p) - F(p).
\]

\[ \blacksquare \]
Note that, for a given value of the parameter, the optimal level can always be taken as $c = 0$ up to $R^+$-equivalence.

## 3.1 Generic cases

Under conditions of Theorem 3.1, the optimal averaged profit $A_l$ for level cycles is the solution of equation $A(p, c) = c$. Besides, due to the last statement of that theorem, we can study the smoothness of $A_l$ by studying the smoothness of the averaged profit $A$ along level cycles.

We present a corollary of the second assertion of Theorem 3.1 and we also include its proof ([5]) because it allow us to recognize all situations where the optimal averaged profit $A_l$ for level cycles can be nonsmooth.

**Corollary 3.4:** Consider a family of pairs of polydynamical systems and profit densities on the circle. When, for a fixed value $p_0$ of the parameter, the optimal averaged profit is provided by a level cycle whose optimal level is a regular value of the profit density then $A_l$ is a differentiable function at $p_0$.

**Proof:** Let $p_0$ be a parameter value and $c_0$ be the corresponding optimal level. If $c_0$ is a regular value of the profit density $f(\cdot, p_0)$ then the number $N$ of solutions of equation $f(x, p_0) = c_0$ is even (because the phase space is the circle). Let $x_1, \cdots, x_N$ be these solutions. By the Implicit Function Theorem, around $(p_0, c_0)$, equation $f(x, p) = c$ has exactly $N$ solutions $x = x_1(p, c), \cdots, x = x_N(p, c)$, where all $x_i$ are smooth functions with $x_i(p_0, c_0) = x_i$.

Without loss of genericity, we assume that $f_x(x_1, p_0) > 0$. Then,

$$T(p, c) = \int_{x_1(p,c)}^{x_2(p,c)} R(x, p) dx + \int_{x_2(p,c)}^{x_1(p,c)} r(x, p) dx + \cdots + \int_{x_{N-1}(p,c)}^{x_N(p,c)} R(x, p) dx + \int_{x_N(p,c)}^{x_1(p,c)} r(x, p) dx,$$

where

$$R(x, p) = \frac{1}{v_{\text{min}}(x, p)} \quad \text{and} \quad r(x, p) = \frac{1}{v_{\text{max}}(x, p)}.$$

Observe that the equality $f(x_i(p, c), p) = c$ implies that

$$\frac{\partial x_i}{\partial c}(p, c) = \frac{1}{f_x(x_i(p, c), p)} \quad \text{and} \quad \frac{\partial x_i}{\partial p_j}(p, c) = -\frac{f_{p_j}(x_i(p, c), p)}{f_x(x_i(p, c), p)}.$$

Then,
\[ T_c(p, c) = -\sum_{i=1}^{\infty} \left( \frac{R - r}{|f_x|} \right) (x_i(p, c), p), \]

\[ T_{pj}(p, c) = \sum_{i=1}^{\infty} \left( \frac{(R - r)f_{pj}}{|f_x|} \right) (x_i(p, c), p) + \int_{x_1(p,c)}^{x_2(p,c)} R_{pj}(x, p) dx + \cdots + \int_{x_1(p,c)}^{x_N(p,c)} r_{pj}(x, p) dx \]

and there are similar formulas for the profit \( P \) (we just must consider the products \( Rf \) and \( rf \) instead of \( R \) and \( r \)). Therefore, \( T \) and \( P \) are \( C^1 \)-functions at \((p_0, c_0)\) because \( R \) and \( r \) are continuous and so, due to Theorem 3.1, we conclude that the optimal averaged profit is a \( C^1 \)-function at \( p_0 \).

Using (3.1) and (3.2) it is possible to identify all situations that can originate, in a generic case, the nonsmoothness of the optimal averaged profit \( A_l \) for level cycles. In fact, we see that \( T \) and \( P \) just can be nonsmooth at points \((p_0, c_0)\) when there are points \((x, p_0)\) where the extremal velocities used are not smooth. This fact justifies the introduction of the following definition.

**Definition 3.5:** Consider a family of polydynamical systems on the circle.

1. The set of points where at least one of the extremal velocities is not smooth is called the Maxwell set.
2. A point where both extremal velocities are not smooth and have different levels is called a self-intersection point of the Maxwell set.
3. A point of the Maxwell set is called tangent for the maximum/minimum velocity if at that point the natural fibration of the product space \( S^1 \times P \) over the parameter space \( P \) is tangent to the set of points where the maximum/minimum velocity is not smooth; otherwise, it is called a regular point for the maximum/minimum velocity.
4. A tangent point for the maximum/minimum velocity has tangency of order \( k \) if \( k \) is the highest order of contact* among all pairs of admissible velocities coinciding at that point and providing that velocity.

**Lemma 3.6:** Consider a 2-parameter family of polydynamical systems on the circle. Generically there are no points of coincidence of more than four admissible velocities.

*Two velocities \( v_1 \) and \( v_2 \) have contact of order \( k \) at a point if

\[ v_1 - v_2 = \cdots = \frac{d^{k-1}}{dx^{k-1}}(v_1 - v_2) = 0 \neq \frac{d^k}{dx^k}(v_1 - v_2) \]

at that point, where \( x \) is a coordinate on the phase space.
Proof: The statement follows immediately from Thom Transversality Theorem. 

Definition 3.7: Consider a 2-parameter family of polydynamical systems on the circle. A point of the Maxwell set is called a double point if one of the extremal velocities is not smooth due to the coincidence of exactly two admissible velocities. Analogously we define triple and quadruple points for the coincidence of exactly three and four admissible velocities, respectively.

The following result identifies all types of tangent points in the Maxwell set.

Lemma 3.8: Consider a 2-parameter family of polydynamical systems on the circle. Generically,

1. the tangency of a double point can be of first or second order
2. the tangency of a triple point just can be of first order
3. a quadruple point is regular.

Proof: The statement follows immediately from Thom Transversality Theorem.

Now that we already know all types of points of the Maxwell set that appear in a generic case, we can understand the following corollary of Corollary 3.4.

Corollary 3.9: Consider a 2-parameter family of polydynamical systems on the circle. Suppose that, for a fixed value $p_0$ of the parameter, the optimal averaged profit is provided by a level cycle whose optimal level is a regular value of the profit density. If the extremal velocities used are smooth on the closure of the domain where they are used or have a finite number of nonsmoothness points only of type “regular double” which are on the interior of such domain then $A_l$ is smooth at $p_0$.

Proof: By (3.1) and (3.2) on page 42, we easily see that in the first situation of this corollary $A_l$ is smooth.

In the second situation the period $T$ and profit $P$ are finite sums of integrals with smooth limits, integrands of which are also smooth except at points of the Maxwell set. But this set is on the interior of the domain of integration and around the point it takes the form $x = X(p)$, with $X$ smooth. So, subdividing the respective interval of integration in two subintervals, one for each side of $x = X(p)$, we can write $T$ and $P$ as finite sums of integrals with smooth limits and smooth integrand functions. Then, they are smooth and so it is the optimal averaged profit $A_l$ for level cycles, due to Theorem 3.1.
In the following theorem we list all situations that have to be considered to analyse all generic singularities of the optimal averaged profit for level cycles.

**Theorem 3.10**: Consider a 2-parameter family of polydynamical systems on the circle. Generically, the optimal averaged profit $A_l$ for level cycles can be nonsmooth only in the following situations:

1. passing through a tangent double point - 1\textsuperscript{st} order tangency
2. passing through a regular triple point
3. switching at a regular double point that is not a self-intersection point of the Maxwell set
4\pm. transition through a local minimum/maximum of the profit density provided by a point out of the Maxwell set
5. passing through a tangent triple point - 1\textsuperscript{st} order tangency
6. passing through a regular quadruple point
7. switching at a regular double point that is a self-intersection point of the Maxwell set
8. switching at a tangent double point - 1\textsuperscript{st} order tangency and $\#U > 2$
9. switching at a tangent double point - 1\textsuperscript{st} order tangency and $\#U = 2$
10. switching at a regular triple point and $\#U > 3$
11\pm. transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U > 2$
12\pm. transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U = 2$
13. switching at a regular triple point and $\#U = 3$
14. passing through a tangent double point - 2\textsuperscript{nd} order tangency
15. transition through a critical value of the profit density which is not a minimum nor a maximum.

**Remark.** In the next section we prove that these situations are in fact the unique situations where the optimal averaged profit $A_l$ for level cycles is nonsmooth. Situations 1-4 have already been studied ([9]) and they correspond to codimension 1 singularities in the parameter space. Situations 5-12 are studied in the first subsection of the next section and their normal forms are presented in Theorem 3.12. Situations 13-15 are studied in the second subsection of the next section and we present a qualitative study for them. Singularities 5-15 correspond to codimension 2 singularities in the parameter space.

**Proof**: By (3.1) and (3.2) on page 42, when the optimal level is a regular value of the profit density then $T$ and $P$ just can be nonsmooth at points $(p_0,c_0)$ when there are points $(x,p_0)$ of the Maxwell set in the following situations:
1. where it is necessary to switch between the maximum and the minimum velocities
2. on the interior of the domain of integration of the extremal velocities used.

Then, there are exactly three situations where the optimal profit for level cycles can be nonsmooth:

**Situation 1.** Passing through a point of the Maxwell set on the interior of the domain where the extremal velocities are used when the optimal level is a regular value of the profit density;

**Situation 2.** Switching (between maximum and minimum velocities) at a point of the Maxwell set when the optimal level is a regular value of the profit density;

**Situation 3.** Transition through a critical point of the profit density having the optimal level.

Using Thom Transversality Theorem we identify all types of points that, in a generic case, have to be included in each one of the previous situations:

1. Situation 1: regular double points, tangent double points with first or second order tangency, regular triple points, tangent triple points with first order tangency and regular quadruple points
2. Situation 2: regular double points which are not self-intersection points, regular double points where both extremal velocities are nonsmooth (self-intersection point), tangent double points with first order tangency and regular triple points
3. Situation 3: points out of the Maxwell set and regular double points which are not self-intersection points.

Finally, due to Corollary 3.9 we have to exclude regular double points from situation 1. □

### 3.2 Singularities of the optimal averaged profit for level cycles

In this section we study all generic singularities of the optimal averaged profit for level cycles.

We firstly analyse point singularities, that is, singularities that arise when for a fixed value of the parameter there is exactly one point leading to a nonsmoothness situation listed on Theorem 3.10. After that, we will use these singularities and Multijet Transversality Theorem to study multi-point singularities, that is, singularities that arise when for a fixed value of the parameter there are at least two points leading to a nonsmoothness situation listed on Theorem 3.10.
3.2.1 Point singularities

3.2.1.1 Singularities 1-12±

Lemma 3.11: Let $F$ and $A$ be smooth functions with $F(0) \neq 0$ and $A(0) = 0$. Equation $c \cdot F(p, c) = A(p)$ has a unique solution $c = C(p)$ satisfying $C(0) = 0$. Moreover, $C(p) = A(p) \cdot B(p)$, for some smooth function $B$ with $B(0) \neq 0$.

Proof: The first part of this lemma follows immediately by the Implicit Function Theorem. Therefore, around the origin,

$$C(p) \cdot F(p, C(p)) = A(p),$$

and so we conclude that $C(p) = A(p) \cdot B(p)$, where $B(p) = \frac{1}{F(p, C(p))}$. Note that $B$ is well defined around the origin because $F(0) \neq 0$. Note, also, that $B$ and $F$ have the same sign around the origin.

Theorem 3.12: Consider a $k$-parameter family of pairs of polydynamical systems and profit densities on the circle and a parameter value for which there is exactly one point leading to a nonsmoothness situation listed on Theorem 3.10. If such situation is not one of the last three of that theorem then, generically, at this parameter value the germ of the optimal averaged profit for level cycles is, up to the equivalence pointed out in the third column, the germ at the origin of one of the functions from the second column of:

- Table 3.1, if $k = 1$,
- Tables 3.1 and 3.2, if $k = 2$.

Table 3.1:

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0, $p_1 \leq 0$ \n $p_1^{3/2}, p_1 \geq 0$</td>
<td>$\Gamma$</td>
</tr>
<tr>
<td>2</td>
<td>$p_1^2, p_1 \geq 0$ \n $p_1^3, p_1 \leq 0$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>3</td>
<td>$p_1^2, p_1 \geq 0$ \n $p_1^3, p_1 \leq 0$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>4±</td>
<td>$p_1^{3/2} \pm p_1^2, p_1 \geq 0$ \n $0, p_1 \leq 0$</td>
<td>$\Gamma$</td>
</tr>
</tbody>
</table>

Remark. All singularities from Table 3.1 are already known [9]. In singularities 5, 8, 11± and 12± of Table 3.2, $H \equiv h(p_1^{3/2}, p)$, where $h$ is a smooth function. Besides, functions $A, B$
3.2. Singularities of the optimal averaged profit for level cycles

Table 3.2:

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0, (2p_0^{3/2},) (p_1^{3/2} - \frac{3}{7}p_1p_2 + \frac{1}{2}p_2^3 + (p_1 - p_2)^2)H,) (</td>
<td>p_2</td>
</tr>
<tr>
<td>6</td>
<td>0, (p_1 \leq 0, p_2 \leq 0) (p_1^2,) (p_1 \geq 0, p_2 \leq p_1) (p_2^2A,) (p_2 \geq 0, p_2 \geq p_1M) (p_1^2 + (p_1 - p_2)^2B,) (p_1 \leq p_2 \leq p_1M)</td>
<td>(\Gamma)</td>
</tr>
<tr>
<td>7</td>
<td>0, (p_1 \leq 0, p_2 \leq 0) (p_1^3,) (p_1 \geq 0, p_2 \leq p_1^3) (p_2^3A,) (p_2 \geq 0, p_1 \leq -p_2^3) (p_1^3 + (p_2 - p_1^3)^3B,) (p_1 \geq -p_2^3, p_2 \geq p_1^3)</td>
<td>(\Gamma)</td>
</tr>
<tr>
<td>8</td>
<td>0, (2(p_1 + p_2)(p_1^{3/2} + p_1^3A),) ((p_1 + p_2)(p_1^{1/2} + p_2)(p_1 + p_1^{5/2}) + (p_1 - p_2)^3)H,) (</td>
<td>p_2</td>
</tr>
<tr>
<td>9</td>
<td>0, ((p_1 + p_2)(p_1^{3/2} + p_1^3A),) ((p_1 + p_2)(-p_1^{3/2} + p_1^3A),) (3p_1^2p_2 + p_1^2p_2^3 + p_2^3 - p_1p_2^3 + 2(p_1 + p_2)p_1^3A + (p_1 - p_2^3)H,) (</td>
<td>p_2</td>
</tr>
<tr>
<td>10</td>
<td>0, (p_1 \leq 0, p_2 \leq 0) (p_1^3,) (p_1 \geq 0, p_2 \leq p_1) (p_2^3A,) (p_2 \geq 0, p_2 \geq p_1M) (p_1^3 + (p_1 - p_2)^3B,) (p_1M \leq p_2 \leq p_1)</td>
<td>(\Gamma)</td>
</tr>
<tr>
<td>11±</td>
<td>0, ((p_1^{3/2} \pm p_1^2),) ((p_1^{3/2}A \pm p_1^2B),) (p_1^{3/2} + p_1^2 + (p_2 - (p_1 + \sqrt{p_1}M))H,) (-(p_1 + \sqrt{p_1}M) \leq p_2 \leq p_1 + \sqrt{p_1}M)</td>
<td>(\Gamma)</td>
</tr>
<tr>
<td>12±</td>
<td>0, ((p_1^{3/2} \pm p_1^2)(p_2 + p_1A),) ((-p_1^{3/2} \pm p_1^2)(p_2 + p_1A),) ((p_1p_2 + p_1^2)(p_2 + p_1A) + (p_1 - p_2^2)H,) (</td>
<td>p_2</td>
</tr>
</tbody>
</table>

and \(M\) are smooth functions of the parameter, with \(M(0) > 1\) in singularities 6 and 10, and \(M(0) = 1\) in singularities 12±.

**Proof:** Consider a parameter value \(p_0\) for which there is exactly one point leading to a nonsmoothness situation listed on Theorem 3.10 and let \(c_0\) be the respective optimal level. By \(R^+\)-equivalence we assume that \(c_0 = 0\).

We will sequentially suppose that for this fixed value of the parameter there is exactly one point in one of situations from Theorem 3.10. We shift this point to the origin. All the calculations presented in this proof are done around this point.
Singularity 1. Passing through a tangent double point with first order tangency

Suppose that the origin is inside the domain where one of the extremal velocities is used and that it is a tangent double point with first order tangency for that velocity. Let \( v_1 \) and \( v_2 \) be the respective velocities that coincide at the origin.

In this situation there are two conditions, namely, \( v_1 - v_2 = (v_1 - v_2)_x = 0 \) at the origin. Besides, \( (v_1 - v_2)_{xx}(0) \neq 0 \) and so, Mather Division Theorem implies that around the origin equation \( (v_1 - v_2)(x, p) = 0 \) is equivalent to \( x^2 + r_1(p)x + r_2(p) = 0 \), where all \( r_i \) are smooth functions vanishing at the origin. Transversality theorems imply that generically \( (v_1 - v_2)p_i(0) \neq 0 \) for some \( i \). Without loss of generality, we suppose that \( (v_1 - v_2)p_i(0) \neq 0 \). Considering new coordinates \( \tilde{x} = x + r_1(p)/2 \) and \( \tilde{p}_1 = r_1^2(p)/4 - r_2(p) \) we reduce, near the origin, the set where the extremal velocity used is not smooth to the form \( x^2 - p_1 = 0 \). Renumbering if necessary the velocities we assume that for all points \( (x, p) \) in a neighborhood of the origin the extremal velocity used is \( v_1 \) if \( p_1 \leq x^2 \) and is \( v_2 \) if \( p_1 \geq x^2 \).

Consequently, around the origin the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & p_1 \leq 0 \\
T_1(p, c) + T_2(p), & p_1 \geq 0
\end{cases}
\]

where \( T_1 \) is the period when around the origin we just use velocity \( v_1 \) and

\[
T_2(p) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right)(x, p)dx = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x^2 - p_1)a(x, p)dx,
\]

where \( a \) is a smooth function that does not vanish at the origin (because we have first order tangency) and \( (af)(0) < 0 \). In fact, if the extremal velocity used is the maximal then \( f(0) < 0 \) (note that \( f \) does not vanish at the origin because it is not a switching point) and \( a(0) > 0 \); in the other situation, \( f(0) > 0 \) and \( a(0) < 0 \).

Analogously we obtain the profit \( P \) around the origin. For \( p_1 \leq 0 \) the optimal averaged profit is the unique solution vanishing at the origin of equation \( c = (P_1/T_1)(p, c) \). Due to the Implicit Function Theorem this equation has a unique solution \( c = c_1(p) \), where \( c_1 \) is
a smooth function defined around the origin. Therefore, subtracting this function from the family of profit densities, which is allowed by $R^+$-equivalence, the solution of the first equation is $c_1(p) = 0$, valid for $p_1 \leq 0$, and the profit $P_1$ is written as $c^2 h(p, c)$, for some smooth function $h$, because $P_1(p, 0) = 0$ and, due to the optimality of the 0-level cycle, $P_{1,c}(p, 0) = 0$. So, for $p_1 \geq 0$ the optimal averaged profit is the unique solution vanishing at the origin of equation

$$
c \left[ (T - ch)(p, c) + \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x^2 - p_1) a(x, p) dx \right] = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x^2 - p_1)(af)(x, p) dx.
$$

Simple calculations reduce this equation to the form

$$
c[(T_1 - ch)(p, c) + p_1^{3/2} \tilde{A}(p)] = p_1^{3/2} A(p),
$$

for some smooth functions $\tilde{A}$ and $A$ with $A(0) = -\frac{4}{3}(af)(0) > 0$. Therefore, its solution is written as $c(p) = p_1^{3/2} C(p_1^{3/2}, p)$, for some smooth function $C$ positive at the origin. Finally, after writing $C(p_1^{3/2}, p) = A(p) + p_1^{3/2} B(p)$, for some smooth functions $A$ and $B$ with $A(0) > 0$, we conclude that this solution takes the form

$$
c(p) = p_1^{3/2} A(p) + p_1^3 B(p)
$$

and so, considering a new coordinate $\tilde{c}$ such that $c = \tilde{c} A(p) + \tilde{c}^2 B(p)$ we get singularity 1.

**Figure 3.1: Singularity 1**

**Singularity 2.** Passing through a regular triple point

Suppose that the origin is inside the domain where one of the extremal velocities is used and that it is a regular triple point for that velocity. Let $v_1$, $v_2$ and $v_3$ be the respective velocities that coincide at the origin.

In this situation there are two conditions, namely, $v_1 - v_2 = v_1 - v_3 = 0$ at the origin. Besides, $(v_i - v_j)_x(0) \neq 0$, for $1 \leq i < j \leq 3$ and so, Mather Division Theorem implies...
the following equivalences around the origin

\[(v_1 - v_2)(x, p) = 0 \Leftrightarrow x + r_1(p) = 0\]
\[(v_2 - v_3)(x, p) = 0 \Leftrightarrow x + r_2(p) = 0\]
\[(v_1 - v_3)(x, p) = 0 \Leftrightarrow x + r_3(p) = 0\]

where all \(r_i\) are smooth functions vanishing at the origin. Transversality theorems imply that generically \((r_2 - r_1)_{p_i}(0) \neq 0\) for some \(i\). Without loss of generality, we assume that \((r_2 - r_1)_{p_i}(0) \neq 0\). Considering new coordinates \(\tilde{x} = x + r_1(p)\) and \(\tilde{p}_1 = (r_1 - r_2)(p)\) we write around the origin

\[(v_1 - v_2)(x, p) = x \cdot V_1(x, p)\]
\[(v_2 - v_3)(x, p) = (x - p_1) \cdot V_2(x, p)\]
\[(v_1 - v_3)(x, p) = (x - \alpha(p)) \cdot V_3(x, p)\]

where all \(V_i\) are smooth functions that do not vanish at the origin and \(\alpha\) is a smooth function vanishing at the origin. Without loss of generality, we assume that \(V_2(0) > 0\) (otherwise we consider new coordinates \(\tilde{x} = -x\) and \(\tilde{p}_1 = -p_1\)). It is easy to see that

\[\alpha(p) = p_1 \frac{V_2(0, p)}{V_3(0, p)}\]
\[V_1(0) = \frac{1 - \alpha_{p_1}(0)}{\alpha_{p_1}(0)} \cdot V_2(0)\]
\[V_3(0) = \frac{1}{\alpha_{p_1}(0)} \cdot V_2(0)\]

Therefore, \(\alpha_{p_1}(0) \neq 0\) and \(\alpha_{p_1}(0) \neq 1\) and so we have to consider if \(\alpha_{p_1}(0)\) is negative or positive and, in this last situation, if it is bigger or smaller than 1. Renumbering if necessary the velocities and choosing suitable coordinates we assume that \(\alpha_{p_1}(0)\) is negative. This corresponds to consider that, at the origin, \(V_2\) is positive and both \(V_1\) and \(V_3\) are negative. Now we have to consider which extremal velocity is being used. Both situations are proved in the same manner and so, we just present the proof for the maximum velocity case. Therefore, for all points \((x, p)\) in a neighborhood of the origin, the velocity used is:

\[v_1, \quad \text{if } x \leq 0 \text{ and } x \leq \alpha(p)\]
\[v_2, \quad \text{if } x \geq 0 \text{ and } x \geq p_1\]
\[v_3, \quad \text{if } \alpha(p) \leq x \leq p_1\]

Consequently, around the origin the period of the \(c\)-level cycle is

\[T(p, c) = \begin{cases} 
T_1(p, c), & p_1 \leq 0 \\
T_1(p, c) + T_2(p), & p_1 \geq 0 
\end{cases}\]
3.2. Singularities of the optimal averaged profit for level cycles

where $T_1$ is the period when, around the origin, we switch from $v_1$ to $v_2$ at $x = 0$ and

$$T_2(p) = \int_0^{p_1} \left( \frac{1}{v_3} - \frac{1}{v_1} \right) (x, p) dx + \int_0^{p_1} \left( \frac{1}{v_3} - \frac{1}{v_2} \right) (x, p) dx.$$ 

As in the previous situation, we reduce by $R^+$-equivalence the optimal averaged profit to

$$c_1 = 0 \text{ for } p_1 \leq 0 \text{ and write } P_1(p, c) = c^2 h(p, c), \text{ for some smooth function } h.$$ 

For $p_1 \geq 0$ the optimal averaged profit is the unique solution vanishing at the origin of equation

$$c[(T_1 - ch)(p, c)] = \int_0^{p_1} (x - \alpha(p))(af - ca)(x, p) dx + \int_0^{p_1} (x - p_1)(bf - cb)(x, p) dx,$$

where $a$ and $b$ are smooth functions with $(af)(0) > 0$ and $(bf)(0) < 0$. Simple calculations reduce last equation to the form $c \cdot \varphi(p, c) = p_1^2 A(p)$, for some smooth functions $\varphi$ and $A$, both positive at the origin. Therefore, the solution of such equation is written as $p_1^2 A(p)$, for some new smooth function $A$ positive at the origin. A simple change of the coordinate $p_1$ leads to singularity 2.

![Figure 3.2: Singularity 2](image)

**Singularity 3.** Switching at a regular double point which is not a self-intersection point of the Maxwell set

Suppose that the origin is a regular double point which is not a self-intersection point of the Maxwell set and that $f(0, 0) = 0$, which is a regular value of the profit density $f(\cdot, 0)$. We consider firstly the existence of more than two admissible velocities and so, in a generic case, exactly one of the extremal velocities is not smooth. We suppose that it is the maximum velocity that is not smooth, due to the coincidence of two velocities $v_1$ and $v_2$. The case of nonsmoothness of the minimum velocity is proved in the same manner.

In this situation there are two conditions, namely, $f = 0$ and $v_1 - v_2 = 0$ at the origin. Besides, $f_\perp(0) \neq 0$ and $(v_1 - v_2)_\perp(0) \neq 0$. By classical results on Singularity Theory [6] we can consider a coordinate system around the origin where the family of profit
densities takes the form $x + \beta(p)$, where $\beta$ is a smooth function vanishing at the origin. Due to Mather Division Theorem, around the origin equation $v_1 - v_2 = 0$ is equivalent to $x + r(p) = 0$, where $r$ is a smooth function that does not vanish at the origin. Transversality theorems imply that generically the rank of the following matrix is maximum

$$
\begin{pmatrix}
(v_1 - v_2)_x & (v_1 - v_2)_{p_1} & (v_1 - v_2)_{p_2} \\
f_x & f_{p_1} & f_{p_2}
\end{pmatrix}
$$

and so, it is also maximum the rank of the following matrix

$$
\begin{pmatrix}
0 & (\beta - r)_{p_1} & (\beta - r)_{p_2} \\
1 & \beta_{p_1} & \beta_{p_2}
\end{pmatrix}
$$

Without loss of generality, we may assume that generically $(\beta - r)_{p_1}(0) \neq 0$. Considering new coordinates $\tilde{x} = x + \beta(p)$ and $\tilde{p}_1 = (\beta - r)(p)$ we reduce near the origin the Maxwell set to the form $x = p_1$ and obtain $f(x, p) = x$. Renumbering if necessary the velocities we get that for all points $(x, p)$ in a neighborhood of the origin the maximum velocity used is $v_1$ if $x \leq p_1$ and is $v_2$ if $x \geq p_1$. So, around the origin, the change of velocities is represented in the following diagram

Consequently, around the origin the period of the $c$-level cycle is

$$T(p, c) = \begin{cases} 
T_1(p, c), & c \leq p_1 \\
(T_1 + T_2)(p, c), & c \geq p_1
\end{cases}$$

where $T_1$ is the period when, around the origin, we switch from $v_1$ to $v_{\text{min}}$ at $x = c$ and

$$T_2(p, c) = \int_{p_1}^{c} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx.$$

As in all previous situations, we reduce by $R^+\text{-equivalence}$ the optimal averaged profit to $c_1 = 0$ for $p_1 \leq 0$ and write $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$. In order to preserve the last diagram we consider new coordinates $\tilde{x} = x - c_1(p)$ and $\tilde{p}_1 = p_1 - c_1(p)$, which generically are justified by transversality theorems. For $c \geq p_1$ the optimal averaged profit is the unique solution vanishing at the origin of equation

$$c = \frac{c^2 h(p, c) + \int_{p_1}^{c} x(x - p_1) a(x, p) dx}{T_1(p, c) + \int_{p_1}^{c} (x - p_1) a(x, p) dx},$$
where \( a \) is a smooth function negative at the origin. This equation takes the form

\[
c[(T_1 - ch)(p, c)] = \int_{p_1}^{c}(x - c)(x - p_1)a(x, p)dx.
\]

Simple calculations show that last equation is equivalent to equation \( c \cdot \phi(p, c) = p_1^3A(p) \), for some smooth functions \( \phi \) and \( A \), with \( \phi(0) > 0 \) and \( A(0) < 0 \) at the origin. Therefore, the solution has the form \( p_1^2C(p) \) for some smooth function \( C \) negative at the origin. Rescaling \( p_1 \) we obtain that the solution of this equation is \( c_2(p) = -p_1^3 \) which is valid when \( c_2(p) \geq p_1 \), that is, when \( p_1 \leq 0 \). Adding \( p_1^3 \) to the solution we obtain singularity 3.

\[\text{Figure 3.3: Singularity 3}\]

When \( v_1 \) and \( v_2 \) are the unique admissible velocities of the polydynamical system then both admissible velocities are not smooth at the origin. As above, we can consider a coordinate system where \( f(x, p) = x \) and where the maximum velocity coincides with \( v_1 \) when \( x \leq p_1 \) and with \( v_2 \) when \( x \geq p_1 \). This implies that the minimum velocity coincides with \( v_2 \) when \( x \leq p_1 \) and with \( v_1 \) when \( x \geq p_1 \). Then, the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & c \leq p_1 \\
(T_1 + T_2)(p, c), & c \geq p_1 
\end{cases}
\]

where \( T_1 \) is the period when around the origin we switch from \( v_1 \) to \( v_2 \) at \( x = c \) and from \( v_2 \) to \( v_1 \) at \( x = p_1 \) and

\[
T_2(p, c) = \int_{p_1}^{c} 2 \left( \frac{1}{v_2} - \frac{1}{v_1} \right)(x, p)dx.
\]

Then, the proof of this case is done in the same manner.
3. Level cycles

**Singularities $4_\pm$.** Transition through a local minimum/maximum of the profit density provided by a point out of the Maxwell set

Suppose that both extremal velocities are smooth at the origin and that $f(0,0) = 0$, which is a (relative) minimum of the profit density $f(\cdot,0)$. The case of a minimum provides singularity $4_+$; the case of a maximum provides singularity $4_-$ and is proved in the same manner. By classical results on Singularity Theory [6] we conclude that near the origin the family of profit densities is written as $x^2 + \beta(p)$, where $\beta$ is a smooth function vanishing at the origin.

Transversality theorems imply that generically $\beta_{p_i}(0) \neq 0$, for some $i = 1, 2$. Without loss of generality we suppose that $\beta_{p_i}(0) \neq 0$ and so, considering $\tilde{p}_1 = -\beta(p)$, $f$ takes the form $x^2 - p_1$. Therefore, around the origin, the change of velocities is represented in the following diagram

Consequently, around the origin the period of the $c$-level cycle is

$$T(p,c) = \begin{cases} T_1(p,c), & c + p_1 \leq 0 \\ T_1(p,c) + T_2(p,c), & c + p_1 \geq 0 \end{cases}$$

where $T_1$ is the period when around the origin we just use the minimum velocity and

$$T_2(p,c) = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} \left( \frac{1}{v_{max}} - \frac{1}{v_{min}} \right) (x,p)dx.$$ 

As in all previous situations, we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p_1 \leq 0$ and write $P_1(p,c) = \epsilon^2 h(p,c)$, for some smooth function $h$. In order to preserve the last diagram we consider a new coordinate $\tilde{p}_1 = p_1 + c_1(p)$, which generically is justified by transversality theorems. For $c + p_1 \geq 0$ the optimal averaged profit is the unique solution vanishing at the origin of equation

$$c[(T_1 - ch)(p,c)] = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} (x^2 - (c + p_1))a(x,p)dx,$$

where $a$ is a smooth function negative at the origin. Simple calculations reduce last equation to the form

$$c\varphi(p,c) = (c + p_1)^{3/2},$$
for some smooth function \( \varphi \) positive at the origin. When \( p_1 = 0 \) the solution \( c \) vanishes and so, \( c = p_1 H(p) \), for some function \( H \). Therefore, replacing this form in last equation we obtain

\[
H\varphi(p, H) = p_1^{1/2},
\]

for a new smooth function \( \varphi \) positive at the origin. So, the solution of this last equation takes the form

\[
c(p) = p_1^{3/2} A(p) + p_1^{1/2} B(p),
\]

where all the functions are smooth and positive at the origin. Then, choosing new coordinates \( \tilde{p}_1 = p_1 (B/A)^2(p) \) and \( \tilde{c} = c(B^3/A^4)(p) \) which preserve the solution of the first equation, we obtain for \( p_1 \geq 0 \) the solution \( c_2 = p_1^{3/2} + p_1^2 \).

\[\begin{array}{c}
\text{Figure 3.4: Singularities 4}_+ \\
\end{array}\]

**Singularity 5.** Passing through a tangent triple point with first order tangency

Suppose that the origin is, simultaneously, inside the domain where one of the extremal velocities is used and a tangent triple point with first order tangency for that velocity. Let \( v_1, v_2 \) and \( v_3 \) be the respective velocities that coincide at the origin.

In this situation there are three conditions, namely, \( v_1 - v_2 = v_1 - v_3 = (v_i - v_j)_x = 0 \) at the origin, for exactly one pair \((i, j)\) with \( 1 \leq i < j \leq 3 \). Renumbering if necessary the velocities we assume that \( (v_1 - v_2)_x(0) = 0 \). Besides, \( (v_1 - v_2)_{xx}(0) \neq 0 \neq (v_2 - v_3)_x(0) \) (which also implies that \( (v_1 - v_3)_x(0) \neq 0 \)) and so, Mather Division Theorem implies the following equivalences around the origin

\[
(v_1 - v_2)(x, p) = 0 \iff x^2 + r_1(p)x + r_2(p) = 0 \\
(v_2 - v_3)(x, p) = 0 \iff x + r_3(p) = 0 \\
(v_1 - v_3)(x, p) = 0 \iff x + r_4(p) = 0
\]

where all \( r_i \) are smooth functions vanishing at the origin. Transversality theorems imply that generically the rank of the matrix

\[
\begin{pmatrix}
0 & r_{2,p_1} & r_{2,p_2} \\
2 & r_{1,p_1} & r_{1,p_2} \\
1 & r_{3,p_1} & r_{3,p_2}
\end{pmatrix}(0)
\]
is maximum and so, it is also maximum the rank of the matrix

\[
\begin{pmatrix}
 r_{2,p_1} \\
 \left(\frac{r_1}{2} - r_3\right)_{p_1} \\
 \left(\frac{r_2}{2} - r_3\right)_{p_2}
\end{pmatrix}(0).
\]

Then, choosing new coordinates \(\tilde{x} = x + \left(\frac{r_1}{2}\right)(p)\), \(\tilde{p}_1 = \left(\frac{r_2}{4} - r_2\right)(p)\) and \(\tilde{p}_2 = \left(\frac{r_2}{4} - r_3\right)(p)\)
we obtain the following equivalences around the origin

\[
(v_1 - v_2)(x,p) = (x^2 - p_1) \cdot V_1(x,p) \tag{3.3}
\]

\[
(v_2 - v_3)(x,p) = (x - p_2) \cdot V_2(x,p)
\]

\[
(v_1 - v_3)(x,p) = (x - \alpha(p)) \cdot V_3(x,p)
\]

where all \(V_i\) are smooth functions that do not vanish at the origin and \(\alpha\) is a smooth function vanishing at the origin. We assume that \(V_2\) is positive at the origin. It is easy to see that

\[
\alpha(p) = p_1 \frac{V_1}{V_3}(0,p) + p_2 \frac{V_2}{V_3}(0,p), \quad V_1(0) = \alpha_{p_1}(0) \cdot V_2(0), \quad V_3(0) = V_2(0), \quad \alpha_{p_2}(0) = 1.
\]

Therefore \(\alpha_{p_1}(0)\) does not vanish and we have to consider both possible signs of this value. Renumbering if necessary the velocities and choosing suitable coordinates we assume that it is negative. This corresponds to consider that, at the origin, \(V_1\) is negative and \(V_2\) and \(V_3\) are positive. Now we have to consider which extremal velocity is being used. Both situations are proved in the same manner and so, we just present the proof for the maximum velocity case. Then, for all points \((x,p)\) in a neighborhood of the origin, the velocity used is

\[
v_1, \quad \text{if } x^2 \leq p_1 \text{ and } x \geq \alpha(p)
\]

\[
v_2, \quad \text{if } x^2 \geq p_1 \text{ and } x \geq p_2
\]

\[
v_3, \quad \text{if } x \leq \alpha(p) \text{ and } x \leq p_2.
\]

In order to understand geometrically this situation note that around the origin the following equivalences hold

\[
p_2 > \alpha(p) \iff p_2^2 < p_1 \quad \text{and} \quad p_2 < \alpha(p) \iff p_2^2 > p_1.
\]

In fact, considering \(x = p_2\) in (3.3) we obtain that

\[
(v_1 - v_2)(p_2,p) > 0 \iff (p_2^2 - p_1)V_1(p_2,p) > 0.
\]

The left side of this equivalence can be replaced by \((v_1 - v_3)(p_2,p) > (v_2 - v_3)(p_2,p)\), that is, \((p_2 - \alpha(p))V_3(p_2,p) > 0\). Consequently, because it is the maximum velocity that is being used, \(V_1(p_2,p) < 0\) and \(V_3(p_2,p) > 0\) for all \(p\) sufficiently close to the origin and we obtain

\[
p_2 > \alpha(p) \iff p_2^2 < p_1.
\]
3.2. Singularities of the optimal averaged profit for level cycles

The other equivalence is proved in the same manner.

Consequently, around the origin the period of the $c$-level cycle is

$$T(p,c) = \begin{cases} T_1(p,c), & p \in S_1 = \{p_1 \leq 0\} \cup \{p_2 \geq \sqrt{p_1}\} \\ T_1(p,c) + T_2(p), & p \in S_2 = \{p_2 \leq -\sqrt{p_1}\} \\ T_1(p,c) + T_3(p), & p \in S_3 = \{-\sqrt{p_1} \leq p_2 \leq \sqrt{p_1}\} \end{cases}$$

where $T_1$ is the period when, around the origin, we switch from $v_3$ to $v_2$ at $x = p_2$ and

$$T_2(p) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x,p) dx$$

$$T_3(p) = \int_{\alpha(p)}^{p_2} \left( \frac{1}{v_1} - \frac{1}{v_3} \right) (x,p) dx + \int_{p_2}^{\sqrt{p_1}} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x,p) dx.$$ 

Note that at points (except the origin) on the boundary between $S_1$ and $S_2$ the optimal averaged profit has a singularity 1 (passing through a tangent double point with first order tangency) and on the boundary of $S_3$ this profit has a singularity 2 (passing through a regular triple point).

As in all previous situations we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p \in S_1$ and write $P_1(p,c) = c^2 h(p,c)$, for some smooth function $h$. For $p \in S_2$ and $p \in S_3$ the optimal averaged profit is the unique solution vanishing at the origin of equations:
\begin{align*}
(2) \ c[(T_1 - ch)(p, c)] &= \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x^2 - p_1)[(af - ca)(x, p)]dx \\
(3) \ c[(T_1 - ch)(p, c)] &= \int_{\alpha(p)}^{\alpha} (x - \alpha(p))[bf - cb](x, p)dx + \int_{p_2}^{p_2} (x^2 - p_1)[(af - ca)(x, p)]dx
\end{align*}
respectively, where \( a \) and \( b \) are smooth functions with \((af)(0) < 0\) and \((bf)(0) > 0\).

We have already seen in the study of the previous singularity 1 that under a suitable coordinate change the solution of equation (2) takes the form \(2p_1^{3/2}\). Using the continuity of the optimal averaged profit function, the solution of equation (3) takes the form

\[ p_1^{3/2} - p_1p_2 + (p_1 - p_2^2)H(p_1^{3/2}, p), \]

for some smooth function \( H \). In fact, when \( p_2 = p_1^{1/2} \), the solution \( c_3 \) must to coincide with \( c_1 \), that is, \( c_3(p_1, p_1^{1/2}) = c_1(p_1, p_1^{1/2}) = 0 \). Hence,

\[ c_3(p) = (p_1^{1/2} - p_2)H_1, \]

for some function \( H_1 \). By a similar reasoning,

\[ c_3(p) = 2p_1^{3/2} + (p_1^{1/2} + p_2)H_2, \]

for some function \( H_2 \). So,

\[ (p_1^{1/2} - p_2)H_1 = 2p_1^{3/2} + (p_1^{1/2} + p_2)H_2 \]

and, consequently,

\[ p_1^{1/2}(H_1 - 2p_1 - H_2) = p_2(H_1 + H_2) = p_1^{1/2}p_2\Gamma, \]

for some function \( \Gamma \). Then,

\[ H_1 = p_1 + (p_1^{1/2} + p_2) \cdot \Gamma/2 \quad \text{and} \quad c_3(p) = p_1^{3/2} - p_1p_2 + (p_1 - p_2^2)H, \]

for some function \( H \). Now, because equation (3) can be written as

\[ c \cdot \varphi(p, p_1^{3/2}) = p_1^{3/2}A(p) + B(p), \]

for some smooth functions \( \varphi \), \( A \) and \( B \), we conclude that \( c_3 \) is a smooth function of \( p \) and \( p_1^{3/2} \) and so it does the function \( H \). Moreover, using the differentiability of the optimal averaged function, the solution of equation (3) takes the form

\[ p_1^{3/2} - 3\frac{3}{2}p_1p_2 + \frac{1}{2}p_2^3 + (p_1 - p_2^2)^2H(p_1^{3/2}, p), \]

for some new smooth function \( H \) and so, we obtain singularity 5.
3.2. Singularities of the optimal averaged profit for level cycles

Figure 3.5: Singularity 5

Singularity 6. Passing through a regular quadruple point

Suppose that the origin is, simultaneously, inside the domain where one of the extremal velocities is used and a regular quadruple point for that velocity. Let \( v_1, v_2, v_3 \) and \( v_4 \) be the respective velocities that coincide at the origin.

In this situation there are three conditions, namely, \( v_1 - v_2 = v_1 - v_3 = v_1 - v_4 = 0 \) at the origin. Besides, \( (v_i - v_j)_x(0) \neq 0 \), for \( 1 \leq i < j \leq 4 \) and then Mather Division Theorem implies the following equivalences around the origin

\[
\begin{align*}
(v_1 - v_2)(x, p) &= 0 \iff x + r_1(p) = 0 \\
(v_1 - v_3)(x, p) &= 0 \iff x + r_2(p) = 0 \\
(v_1 - v_4)(x, p) &= 0 \iff x + r_3(p) = 0 \\
(v_2 - v_3)(x, p) &= 0 \iff x + r_4(p) = 0 \\
(v_2 - v_4)(x, p) &= 0 \iff x + r_5(p) = 0 \\
(v_3 - v_4)(x, p) &= 0 \iff x + r_6(p) = 0
\end{align*}
\]

where all \( r_i \) are smooth functions vanishing at the origin. Transversality theorems imply that generically the rank of the following matrix is maximum

\[
\begin{pmatrix}
1 & r_{1,p_1} & r_{1,p_2} \\
1 & r_{2,p_1} & r_{2,p_2} \\
1 & r_{3,p_1} & r_{3,p_2}
\end{pmatrix}(0).
\]

Choosing new coordinates \( \tilde{x} = x + r_1(p) \), \( \tilde{p}_1 = (r_1 - r_2)(p) \) and \( \tilde{p}_2 = (r_1 - r_3)(p) \) we write around the origin

\[
\begin{align*}
(v_1 - v_2)(x, p) &= x \cdot V_1(x, p) \\
(v_1 - v_3)(x, p) &= (x - p_1) \cdot V_2(x, p) \\
(v_1 - v_4)(x, p) &= (x - p_2) \cdot V_3(x, p) \\
(v_2 - v_3)(x, p) &= (x - \alpha(p)) \cdot V_4(x, p) \\
(v_2 - v_4)(x, p) &= (x - \beta(p)) \cdot V_5(x, p) \\
(v_3 - v_4)(x, p) &= (x - \gamma(p)) \cdot V_6(x, p)
\end{align*}
\]
where all functions are smooth with \( V_i \neq 0 \) and \( \alpha = \beta = \gamma = 0 \) at the origin. We assume that \( V_2 \) is negative at the origin. It is easy to show that

\[
\alpha(p) = p_1 \frac{V_2}{V_4} (0, p), \quad \beta(p) = p_2 \frac{V_3}{V_5} (0, p), \quad \gamma(p) = -p_1 \frac{V_2}{V_6} (0, p) + p_2 \frac{V_3}{V_6} (0, p),
\]

\[
V_4(0) = (V_2 - V_1)(0), \quad V_5(0) = (V_3 - V_1)(0), \quad V_6(0) = (V_3 - V_2)(0).
\]

Then, \( \alpha_{p_1}(0), \beta_{p_2}(0) \) and \((\alpha_{p_1} - \beta_{p_2})(0)\) do not vanish and we have to consider both possible signs of these values. Renumbering if necessary the velocities and choosing suitable coordinates we assume that they are negative. This corresponds to consider that, at the origin, \( V_1, V_2 \) and \( V_3 \) are negative and \( V_4, V_5 \) and \( V_6 \) are positive. Now, we have to consider which extremal velocity is being used. Both situations are proved in the same manner and so we just present the proof for the maximum velocity case. Then, for all points \((x, p)\) in a neighborhood of the origin, the velocity used is

\[
v_1, \quad \text{if } x \leq \min\{0, p_1, p_2\},
\]

\[
v_2, \quad \text{if } x \geq \max\{0, \alpha(p), \beta(p)\},
\]

\[
v_3, \quad \text{if } x \geq \max\{p_1, \gamma(p)\} \text{ and } x \leq \alpha(p),
\]

\[
v_4, \quad \text{if } x \geq p_2 \text{ and } x \leq \min\{\beta(p), \gamma(p)\}.
\]

In order to understand geometrically this case, note that around the origin the following equivalences hold

\[
\alpha(p) > 0 \iff p_1 < 0, \quad \beta(p) > 0 \iff p_2 < 0,
\]

\[
\gamma(p) > p_1 \iff \gamma(p) > p_2 \iff p_2 < p_1, \quad \gamma(p) > \alpha(p) \iff \gamma(p) > \beta(p) \iff \alpha(p) < \beta(p),
\]

as well as the similar equivalences that are obtained replacing “>” by “<” and vice-versa.

Notice that \( \beta(p) = \gamma(p) \) if and only if \( p_2 = M(p_1)p_1 \), for some smooth function \( M \) with
3.2. Singularities of the optimal averaged profit for level cycles

\[ M(0) > 1. \] Consequently, around the origin the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & p \in S_1 = \{ p_1 \geq 0, p_2 \geq 0 \} \\
T_1(p, c) + T_2(p), & p \in S_2 = \{ p_1 \leq 0, p_1 \leq p_2 \} \\
T_1(p, c) + T_3(p), & p \in S_3 = \{ p_2 \leq 0, p_2 \leq M(p_1)p_1 \} \\
T_1(p, c) + T_4(p), & p \in S_4 = \{ p_2 \geq M(p_1)p_1, p_2 \leq p_1 \}
\end{cases}
\]

where \( T_1 \) is the period when, around the origin, we switch from \( v_1 \) to \( v_2 \) at \( x = 0 \) and

\[
T_2(p) = \int_{p_1}^{0} \left( \frac{1}{v_3} - \frac{1}{v_1} \right) (x, p) dx + \int_{0}^{\alpha(p)} \left( \frac{1}{v_3} - \frac{1}{v_2} \right) (x, p) dx
\]

\[
T_3(p) = \int_{p_2}^{0} \left( \frac{1}{v_4} - \frac{1}{v_1} \right) (x, p) dx + \int_{0}^{\beta(p)} \left( \frac{1}{v_4} - \frac{1}{v_2} \right) (x, p) dx
\]

\[
T_4(p) = \int_{0}^{\alpha(p)} \left( \frac{1}{v_3} - \frac{1}{v_2} \right) (x, p) dx + \int_{p_2}^{0} \left( \frac{1}{v_4} - \frac{1}{v_1} \right) (x, p) dx + \int_{0}^{\gamma(p)} \left( \frac{1}{v_4} - \frac{1}{v_3} \right) (x, p) dx.
\]

Note that at points (except the origin) on the boundary between these situations the optimal averaged profit has a singularity 2.

As in all previous situations we reduce by \( R^+ \)-equivalence the optimal averaged profit to \( c_1 = 0 \) for \( p \in S_1 \) and write \( P_1(p, c) = c^2 h(p, c) \), for some smooth function \( h \). For \( p \in S_2 \), \( p \in S_3 \) and \( p \in S_4 \) the optimal averaged profit is the unique solution vanishing at the origin of equations:

\[
(2) \ c[(T_1 - ch)(p, c)] = \int_{p_1}^{0} (x - p_1)[(af - ca)(x, p)]dx + \int_{0}^{\alpha(p)} (x - \alpha(p))[bf - cb](x, p)]dx
\]

\[
(3) \ c[(T_1 - ch)(p, c)] = \int_{p_2}^{0} (x - p_2)[ef - ce](x, p)]dx + \int_{0}^{\beta(p)} (x - \beta(p))[gf - cg](x, p)]dx
\]
(4) \( c[(T_1 - ch)(p, c)] = \int_0^{\alpha(p)} (x - \alpha(p))[(bf - cb)(x, p)]dx + \int_{p_2}^0 (x - p_2)[(ef - ce)(x, p)]dx \)

\[ + \int_0^{\gamma(p)} (x - \gamma(p))[(jf - cj)(x, p)]dx \]

respectively, where \( a, b, c, e, g \) and \( j \) are smooth functions with \( (af)(0) > 0, (bf)(0) < 0, (ef)(0) > 0, (gf)(0) < 0 \) and \( (jf)(0) < 0 \). Simple calculations reduce equations (2) and (3) to the form

\[ (2) \ c \cdot \varphi_1(p, c) = p_1^2 B(p), \quad \text{and} \quad (3) \ c \cdot \varphi_2(p, c) = p_2^2 A(p), \]

respectively, where all the functions are smooth and positive at the origin. Therefore, the respective solutions vanishing at the origin of these equations are of the form

\[ c_2(p) = p_1^2 B(p), \quad p \in S_2 \quad \text{and} \quad c_3(p) = p_2^2 A(p), \quad p \in S_3 \]

for some new smooth functions \( A \) and \( B \) positive at the origin. Function \( B \) is removed \( (B \equiv 1) \) doing \( \tilde{c} = c/B(p) \). Then, using the fact that the optimal averaged profit on the boundaries of all situations is of class \( C^1 \), we conclude that \( c_4(p) = p_1^2 + (p_1 - p_2)^2 B(p) \), for a new smooth function \( B \) that generically does not vanish at the origin and we obtain singularity 6.

**Figure 3.6: Singularity 6**

**Singularity 7.** Switching at a regular double point which is a self-intersection point of the Maxwell set

Suppose that the origin is a regular double point for both extremal velocities which is a self-intersection point of the Maxwell set and that \( f(0, 0) = 0 \), which is a regular value of the profit density \( f(\cdot, 0) \). Observe that there are at least four admissible velocities. In a generic case, around this point, each extremal velocity appears under competition of
exact two admissible velocities, say $v_1$ and $v_2$ for the maximum velocity, and $v_3$ and $v_4$ for the minimum velocity.

In this situation there are three conditions, namely, $f = 0$ and $v_1 - v_2 = v_3 - v_4 = 0$ at the origin. Besides, $f_x(0) \neq 0$ and $(v_1 - v_2)_x(0) \neq (v_3 - v_4)_x(0)$. By classical results on Singularity Theory [6] and Mather Division Theorem, we can consider a coordinate system around the origin where the family of profit densities takes the form $x + \beta(p)$ and

$$(v_1 - v_2)(x, p) = 0 \iff x + r_1(p) = 0,$$

$$(v_3 - v_4)(x, p) = 0 \iff x + r_2(p) = 0,$$

where $\beta$, $r_1$ and $r_2$ are smooth functions vanishing at the origin. Transversality theorems imply that generically

$$\left| \begin{array}{ccc} f_x & f_{p_1} & f_{p_2} \\ 1 & (v_1 - v_2)_p & (v_1 - v_2)_p \\ 1 & (v_3 - v_4)_p & (v_3 - v_4)_p \end{array} \right| (0) = \left| \begin{array}{cc} \beta_{p_1} & \beta_{p_2} \\ r_{1,p_1} & r_{1,p_2} \\ r_{2,p_1} & r_{2,p_2} \end{array} \right| (0) \neq 0.$$

Choosing new coordinates $\tilde{x} = x + \beta(p)$, $\tilde{p_1} = (\beta - r_1)(p)$ and $p_2 = (\beta - r_2)(p)$ we have, near the origin, $f(x, p) = x$ and, without loss of generality, we may suppose that

$$v_{\text{max}}(x, p) = \begin{cases} v_1(x, p), & x \leq p_1 \\ v_2(x, p), & x \geq p_1 \end{cases} \quad \text{and} \quad v_{\text{min}}(x, p) = \begin{cases} v_3(x, p), & x \leq p_2 \\ v_4(x, p), & x \geq p_2 \end{cases}$$

Therefore, around the origin, the change of velocities is represented in the following diagram

Note that at points (except the origin) on the boundary between these situations the optimal averaged profit has a singularity 2.
Consequently, around the origin the period of the $c$-level cycle is

$$T(p, c) = \begin{cases} T_1(p, c), & p \in S_1 = \{p_2 \leq c \leq p_1\} \\ (T_1 + T_2)(p, c), & p \in S_2 = \{c \geq \max\{p_1, p_2\}\} \\ (T_1 + T_3)(p, c), & p \in S_3 = \{c \leq \min\{p_1, p_2\}\} \\ (T_1 + T_2 + T_3)(p, c), & p \in S_4 = \{p_1 \leq c \leq p_2\} \end{cases}$$

where $T_1$ is the period when, around the origin, we switch from $v_1$ to $v_4$ at $x = c$ and

$$T_2(p, c) = \int_{p_1}^c \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx \quad \text{and} \quad T_3(p, c) = \int_c^{p_2} \left( \frac{1}{v_3} - \frac{1}{v_4} \right) (x, p) dx.$$

As in all previous situations we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p \in S_1$ and write $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$. In order to preserve the last diagram we consider new coordinates $\tilde{x} = x - c_1(p)$, $\tilde{p}_1 = p_1 - c_1(p)$ and $\tilde{p}_2 = p_2 - c_1(p)$, which generically are justified by transversality theorems. For $p \in S_2$, $p \in S_3$ and $p \in S_4$ the optimal averaged profit is the unique solution vanishing at the origin of equations

1. $c[\tilde{T}_1(p, c)] = \int_{p_1}^{c} (x - c)(x - p_1) a(x, p) dx$
2. $c[\tilde{T}_1(p, c)] = \int_{p_1}^{c} (x - c)(x - p_2) b(x, p) dx$
3. $c[\tilde{T}_1(p, c)] = \int_{p_1}^{c} (x - c)a(x, p) dx + \int_{c}^{p_2} (x - c)(x - p_2)b(x, p) dx$

respectively, where $a$ and $b$ are smooth functions with $a(0) < 0$ and $b(0) > 0$.

Simple calculations show that the integrals in equations (2) and (3) take the form $p_1^3 M(p)$ and $p_2^3 N(p)$, respectively, where $M$ and $N$ are smooth functions with $M(0) < 0$ and $N(0) > 0$. Therefore, the respective solutions vanishing at the origin of these equations are written as $c_2(p) = -p_1^3 B(p)$ and $c_3(p) = p_2^3 A(p)$, where $A$ and $B$ are smooth functions both positive at the origin, which are valid for $p \in S_2$ and $p \in S_3$, respectively.

Changing the sign of $p_1$ we remove the “$-$” in $c_2$ and obtain

$$S_1 = \{p_1 \leq 0, \ p_2 \leq 0\}$$
$$S_2 = \{p_1 \geq 0, \ p_2 \leq p_1^3 B(p)\}$$
$$S_3 = \{p_2 \geq 0, \ p_1 \leq -p_2^3 A(p)\}.$$

Considering $\tilde{p}_1 = (B^3 A)^{1/8}(p)p_1$, $\tilde{p}_2 = (BA^3)^{1/8}(p)p_2$ and $\tilde{c} = (BA^3)^{1/8}(p)c$ as new coordinates, function $B$ is completely removed and function $A$ is removed from the boundary condition. The normal form for $c_4$ is found using the fact that the optimal averaged profit on the boundaries between all situations is of class $C^2$. 

3. Level cycles
3.2. Singularities of the optimal averaged profit for level cycles

Figure 3.7: Singularity 7

**Singularity 8.** Switching at a tangent double point with first order tangency and \( \#U > 2 \)

Suppose that the origin is a tangent double point with first order tangency for one of the extremal velocities and that \( f(0,0) = 0 \), which is a regular value of the profit density \( f(\cdot,0) \). We consider the existence of more than two admissible velocities and so, in a generic case, exactly one of the extremal velocities is not smooth. We suppose that it is the maximum velocity that is not smooth, due to the coincidence of two velocities \( v_1 \) and \( v_2 \). The case of nonsmoothness of the minimum velocity is proved in the same manner.

In this situation there are three conditions, namely, \( f = 0 \) and \( v_1 - v_2 = (v_1 - v_2)_x = 0 \) at the origin. Besides, \( f_x(0) \neq 0 \neq (v_1 - v_2)_x(0) \). By classical results on Singularity Theory [6] and Mather Division Theorem, we can consider a coordinate system around the origin where the family of profit densities takes the form \( x + \beta(p) \) and

\[
(v_1 - v_2)(x,p) = 0 \iff x^2 + r_1(p)x + r_2(p) = 0,
\]

where \( \beta, r_1 \) and \( r_2 \) are smooth functions vanishing at the origin. Transversality theorems justify that \( \tilde{x} = x + (r_1/2)(p) \), \( \tilde{p}_1 = (r_1^2/4 - r_2)(p) \) and \( \tilde{p}_2 = (r_1/2 - \beta)(p) \) can generically be taken as new coordinates and so we write the family of profit densities and the Maxwell set as \( x - p_2 \) and \( x^2 = p_1 \), respectively. Without loss of generality we assume that for all points \( (x,p) \) in a neighborhood of the origin the maximal velocity coincides with \( v_1 \) if \( p_1 \leq x^2 \) and with \( v_2 \) if \( p_1 \geq x^2 \). Therefore, around the origin, the change of velocities is represented in the following diagram.
Note that at points (except the origin) on the boundary between situations 1 and 2 the optimal averaged profit has a singularity 1 and on the boundary of situation 3 this profit has a singularity 3.

Consequently, around the origin the period of the $c$-level cycle is

$$T(p, c) = \begin{cases} \frac{1}{a}, & p \in S_1 = \{p_1 \leq 0\} \cup \{c + p_2 \leq -\sqrt{p_1}\} \\ (T_1 + T_2)(p, c), & p \in S_2 = \{c + p_2 \geq \sqrt{p_1}\} \\ (T_1 + T_3)(p, c), & p \in S_3 = \{-\sqrt{p_1} \leq c + p_2 \leq \sqrt{p_1}\} \end{cases}$$

where $T_1$ is the period when, around the origin, we switch from $v_1$ to $v_{\min}$ at $x = c + p_2$ and

$$T_2(p, c) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx \quad \text{and} \quad T_3(p, c) = \int_{-\sqrt{p_1}}^{c + p_2} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx.$$
It is now easy to understand that this condition is reduced to the form $p_2 \geq p_1^{1/2}$ if we consider new coordinates of the form $\tilde{p}_2 = p_2 + p_1^3 D(p)$ and $\tilde{p}_1 = p_1 E(p)$, for some smooth functions $D$ and $E$ with $D(0) = 0$ and $E(0) = 1$. In these new coordinates $c_2$ is written as

$$c_2(p) = 2(p_1 A(p) + p_2)(p_1^{3/2} H_1(p) + p_1^3 H_2(p)),$$

for new smooth functions $A$, $H_1$ and $H_2$ with $A(0) = 1$, $H_1(0) > 0$ and, generically, $H_2(0) \neq 0$. Finally, we consider $\tilde{p}_1 = A^2(p)p_1$, $\tilde{p}_2 = A(p)p_2$ and $\tilde{c} = (A^4/H_1)(p)c$ as new coordinates (preserving the boundary of situation 2) and we get the following normal form for our solution

$$c_2(p) = 2(p_1 + p_2)(p_1^{3/2} + p_1^3 A(p)),$$

where $A$ is a new smooth function that generically does not vanish at the origin. Note that all coordinate changes used to simplify the boundary condition of situation 2 and to obtain the normal form for its solution do not change anything related to the first equation. Therefore, the third equation is valid for $|p_2| \leq \sqrt{p_1}$. The normal form for its solution obtained using the continuity of the optimal averaged profit (in this case it is difficult to use differentiability) has the form

$$(p_1 + p_2)(p_1^{1/2} + p_2)(p_1 + p_1^{5/2} A) + (p_1 - p_2^2) H(p_1^{3/2}, p),$$

where $H$ is a smooth function that generically does not vanish at the origin.

**Singularity 9.** Switching at a tangent double point with first order tangency and $\#U = 2$

Suppose that the polydynamical system has exactly two admissible velocities, $v_1$ and $v_2$, the origin is a tangent double point with first order tangency (for both extremal velocities) and that $f(0, 0) = 0$, which is a regular value of the profit density $f(\cdot, 0)$. The change of velocities is represented in the following diagram
Note that at points (except the origin) on the boundary of situation 1 the optimal averaged profit has a singularity 1 and on the boundary of situation 4 the profit has a singularity 3.

Consequently, around the origin the period of the $c$-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & p \in S_1 = \{p_1 \leq 0\} \\
(T_1 + T_2)(p, c), & p \in S_2 = \{c + p_2 \leq -\sqrt{p_1}\} \\
(T_1 + T_3)(p, c), & p \in S_3 = \{c + p_2 \geq \sqrt{p_1}\} \\
(T_1 + T_4)(p, c), & p \in S_4 = \{-\sqrt{p_1} \leq c + p_2 \leq \sqrt{p_1}\}
\end{cases}
\]

where $T_1$ is the period when, around the origin, we switch from $v_1$ to $v_2$ at $x = c + p_2$ and

\[
T_2(p, c) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx
\]

\[
T_3(p, c) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx = -T_2(p, c)
\]

\[
T_4(p, c) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx + \int_{\sqrt{c + p_2}}^{\sqrt{p_1}} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx.
\]

As in all previous situations we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p \in S_1$ and write $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$. In order to preserve the last diagram we consider a new coordinate $\tilde{p}_2 = p_2 + c_1(p)$, which generically is justified by transversality theorems. For $p \in S_2$ and $p \in S_3$ the optimal averaged profit is the unique solution vanishing at the origin of equations (2\textendash) and (2\textdagger), respectively, with

\[
(2\textdagger)_{\pm} c[(T_1 - ch)(p, c)] = \pm \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x - (c + p_2))(a^2 - p_1)a(x, p) dx,
\]
where \( a \) is a smooth function negative at the origin. As in the previous situation, these equations are written as

\[
c \cdot \varphi(p, c) = \pm 2p_1^{3/2}[(p_2 + c)A(p, c) + p_1B(p)],
\]

for some smooth functions \( \varphi, A \) and \( B \) with \( \varphi(0) > 0, A(0) > 0 \) and generically \( B(0) \neq 0 \). Therefore, both equations \((2_\pm)\) have the form

\[
c \cdot [\varphi(p, c) + qD(p, c)] = 2q[p_2A(p) + p_1B(p)],
\]

where \( D \) is a smooth function. This fact implies that both solutions take the form

\[
c(p) = q(p_2A(p) + p_1B(p))H(q, p),
\]

for some smooth function \( H \) positive at the origin. Then, we obtain solutions

\[
c_\pm(p) = \pm p_1^{3/2}(p_2A(p) + p_1B(p))[H_1(p) \pm p_1^{3/2}H_2(p)], \quad \pm(c_\pm + p_2) \geq \sqrt{p_1}
\]

where all functions are smooth with \( A(0) > 0, H_1(0) > 0 \) and, generically, \( B(0) \neq 0 \neq H_2(0) \). Considering the same coordinate changes that were used before to obtain the normal form for the second equation, we obtain the following normal forms

\[
c_\pm(p) = (p_1 + p_2)(\pm p_1^{3/2} + p_1^2A(p)),
\]

for \( \pm p_2 \geq \sqrt{p_1} \), where \( A \) is a smooth function that generically does not vanish at the origin. Note that all coordinate changes used to obtain these normal forms do not change anything related to the first equation. Therefore, the fourth equation is valid for \( |p_2| \leq \sqrt{p_1} \). Using the fact that the optimal averaged profit on the boundary of situation 4 is of class \( C^1 \) (using the fact that this profit is of class \( C^2 \) on that boundary leads to a more complex normal form) we obtain the following normal form for its solution

\[
3p_1^2p_2 + p_1p_2^2 + p_1^2 - p_1p_2^3 + 2(p_1 + p_2)p_1^3A + (p_1 - p_2^2)^2H(p_1^{3/2}, p),
\]

where \( H \) is a smooth function that does not vanish at the origin.

Figure 3.9: Singularity 9
Singularity 10. Switching at a regular triple point and \#U > 3

Suppose that the origin is a regular triple point for one of the extremal velocities and that \( f(0,0) = 0 \), which is a regular value of the profit density \( f(\cdot,0) \). We consider the existence of more than three admissible velocities and so, in a generic case, exactly one of the extremal velocities is not smooth. We suppose that it is the maximum velocity that is not smooth, due to the coincidence of exactly three admissible velocities \( v_1, v_2 \) and \( v_3 \).

The case of nonsmoothness of the minimum velocity is proved in the same manner.

In this situation there are three conditions, namely, \( f = 0 \) and \( v_1 - v_2 = v_2 - v_3 = 0 \) at the origin. Besides, \( f_x(0) \neq 0 \) and \( (v_1 - v_2)_x(0) \neq 0 \). By classical results on Singularity Theory [6] and Mather Division Theorem we can consider a coordinate system around the origin where the family of profit densities takes the form \( x + \beta(p) \) and

\[
\begin{align*}
(v_1 - v_2)(x,p) &= 0 \Leftrightarrow x + r_1(p) = 0 \\
(v_2 - v_3)(x,p) &= 0 \Leftrightarrow x + r_2(p) = 0 \\
(v_1 - v_3)(x,p) &= 0 \Leftrightarrow x + r_3(p) = 0,
\end{align*}
\]

where \( \beta, r_1, r_2 \) and \( r_3 \) are smooth functions vanishing at the origin. Transversality theorems justify that \( \tilde{x} = x + r_1(p) \), \( \tilde{p}_1 = (r_1 - r_2)(p) \) and \( \tilde{p}_2 = (r_1 - \beta)(p) \) can generically be taken as new coordinates and so we obtain \( f(x,p) = x - p_2 \) and

\[
\begin{align*}
(v_1 - v_2)(x,p) &= 0 \Leftrightarrow x = 0 \\
(v_2 - v_3)(x,p) &= 0 \Leftrightarrow x = p_1 \\
(v_1 - v_3)(x,p) &= 0 \Leftrightarrow x = \alpha(p)
\end{align*}
\]

where \( \alpha \) is a smooth function of the form \( M(p)p_1 \). As it was seen above in singularity 2, we can assume that \( M(0) < 0 \) and that for all points \( (x,p) \) in a neighborhood of the origin the maximal velocity coincides with \( v_1 \) if \( x \leq \min\{0, \alpha(p)\} \), with \( v_2 \) if \( x \geq \max\{0, p_1\} \) and with \( v_3 \) if \( \alpha(p) \leq x \leq p_1 \). Therefore, around the origin, the change of velocities is represented in the following diagram
Note that at points (except the origin) on the boundary between situations 2 and 4 the optimal averaged profit has a singularity 2 and on the other boundaries the profit has a singularity 3.

Consequently, around the origin the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & p \in S_1 = \{p_1 \leq 0, c + p_2 \leq 0\} \cup \{p_1 \geq 0, c + p_2 \leq \alpha(p)\} \\
(T_1 + T_2)(p, c), & p \in S_2 = \{p_1 \leq 0, c + p_2 \geq 0\} \\
(T_1 + T_3)(p, c), & p \in S_3 = \{p_1 \geq 0, \alpha(p) \leq c + p_2 \leq p_1\} \\
(T_1 + T_4)(p, c), & p \in S_4 = \{p_1 \geq 0, c + p_2 \geq p_1\}
\end{cases}
\]

where \( T_1 \) is the period when around the origin we change from \( v_1 \) to \( v_{\min} \) at \( x = c + p_2 \) and

\[
T_2(p, c) = \int_0^{c + p_2} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx \\
T_3(p, c) = \int_{\alpha(p)}^{c + p_2} \left( \frac{1}{v_3} - \frac{1}{v_1} \right) (x, p) dx \\
T_4(p, c) = \int_{\alpha(p)}^{p_1} \left( \frac{1}{v_3} - \frac{1}{v_1} \right) (x, p) dx + \int_{p_1}^{c + p_2} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx.
\]

As in all previous situations we reduce by \( R^+ \)-equivalence the optimal averaged profit to \( c_1 = 0 \) for \( p \in S_1 \) and write \( P_1(p, c) = c_2 h(p, c) \), for some smooth function \( h \). In order to preserve the last diagram we consider a new coordinate \( \tilde{p}_2 = p_2 + \alpha(p) \), which generically is justified by transversality theorems. For \( p \in S_2 \) and \( p \in S_3 \) the optimal averaged profit is the unique solution vanishing at the origin of equations

\[
(2)\ c[(T_1 - ch)(p, c)] = \int_0^{c + p_2} x(x - (c + p_2)) a(x, p) dx \\
(3)\ c[(T_1 - ch)(p, c)] = \int_{\alpha(p)}^{} (x - \alpha(p))(x - (c + p_2)) b(x, p) dx,
\]

respectively, where \( a \) and \( b \) are smooth functions negative at the origin. Simple calculations reduce equations (2) and (3) to the form

\[
(2)\ c \cdot \varphi_1(p, c) = p_2^3 A(p), \quad \text{and} \quad (3)\ c \cdot \varphi_2(p, c) = (p_2 - \alpha(p))^3 B(p),
\]

where all functions are smooth and positive at the origin. Therefore, the solutions of these equations are written as

\[
c_2(p) = p_2^3 A(p), \quad p \in S_2 \quad \text{and} \quad c_3(p) = (p_2 - \alpha(p))^3 B(p), \quad p \in S_3
\]
for some new smooth functions $A$ and $B$ positive at the origin. Considering a new coordinate $\tilde{p}_1 = p_2 - \alpha(p)$ we obtain the following solutions

\[
\begin{align*}
    c_1(p) &= 0, & p_1 &\leq 0, p_2 &\leq 0 \\
    c_2(p) &= p_2^3 A(p), & p_1 &\leq p_2, p_2 &\geq 0 \\
    c_3(p) &= p_1^3 B(p), & p_1 &\geq 0, p_2 &\leq p_1 M(p),
\end{align*}
\]

for some new smooth functions $A$, $B$ and $M$ positive at the origin and $0 < M(0) < 1$. After this we choose $\tilde{p}_1 = p_1 M(p)$ and obtain

\[
\begin{align*}
    c_1(p) &= 0, & p_1 &\leq 0, p_2 &\leq 0 \\
    c_2(p) &= p_2^3 A(p), & p_2 &\geq p_1 M(p), p_2 &\geq 0 \\
    c_3(p) &= p_1^3 B(p), & p_1 &\geq 0, p_2 &\leq p_1,
\end{align*}
\]

for some new smooth functions $A$, $B$ and $M$ positive at the origin and $M(0) > 1$. Through a simple change of the coordinate $c$ we remove function $B$.

Finally, we have that the fourth equation is valid for $p_1 M(p) \leq p_2 \leq p_1$. Using the fact that the optimal averaged profit on the boundary between situations 3 and 4 is of is of class $C^2$ we obtain $p_1^3 + (p_1 - p_2)^3 B(p)$ as normal form for the solution of the fourth equation, for some new smooth function $B$ that does not vanish the origin.

**Figure 3.10: Singularity 10**

**Singularities 11±.** Transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U > 2$

Suppose that the origin is a regular double point for one of the extremal velocities and that $f(0,0) = 0$, which is a (relative) minimum of the profit density $f(\cdot, 0)$. The case of a minimum provides singularity 11+; the case of a maximum provides singularity 11− and is proved in the same manner. We consider the existence of more than two admissible velocities and so, in a generic case, exactly one of the extremal velocities is not smooth. We suppose that it is the maximum velocity that is not smooth, due to the coincidence of
exactly two admissible velocities \( v_1 \) and \( v_2 \). The case of nonsmoothness of the minimum velocity is proved in the same manner.

In this situation there are three conditions, namely, \( f = f_x = 0 \) and \( v_1 - v_2 = 0 \) at the origin. It is assumed that \((v_1 - v_2)_x(0) \neq 0\) and by genericity \( f_{xx}(0) \neq 0\). By classical results on Singularity Theory [6] and Mather Division Theorem we can consider a coordinate system around the origin where the family of profit densities takes the form \( x^2 + \beta(p) \) and equation \((v_1 - v_2)(x, p) = 0\) is equivalent to \( x + r(p) = 0\), where \( \beta \) and \( r \) are smooth functions vanishing at the origin. Transversality theorems justify that \( \tilde{p}_1 = -\beta(p) \) and \( \tilde{p}_2 = -r(p) \) can generically be taken as new coordinates and so we write the family of profit densities and the Maxwell set as \( x^2 - p_1 \) and \( x = p_2 \), respectively. Without loss of generality, we assume that for all points \((x, p)\) in a neighborhood of the origin the maximal velocity coincides with \( v_1 \) if \( x \leq p_2 \) and coincides with \( v_2 \) if \( x \geq p_2 \). Therefore, around the origin, the change of velocities is represented in the following diagram:

Note that at points (except the origin) on the boundary of situation 1 the optimal averaged profit has a singularity 4 and on the boundary of situation 4 the profit has a singularity 3.

Consequently, around the origin the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c), & p \in S_1 = \{c + p_1 \leq 0\} \\
(T_1 + T_2)(p, c), & p \in S_2 = \{p_2 \geq \sqrt{c + p_1}\} \\
(T_1 + T_3)(p, c), & p \in S_3 = \{p_2 \leq -\sqrt{c + p_1}\} \\
(T_1 + T_4)(p, c), & p \in S_4 = \{-\sqrt{c + p_1} \leq p_2 \leq \sqrt{c + p_1}\}
\end{cases}
\]
where $T_1$ is the period when around the origin we just use the minimum velocity and

$$T_2(p, c) = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} \left( \frac{1}{v_1} - \frac{1}{v_{\text{min}}} \right) (x, p) dx$$

$$T_3(p, c) = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} \left( \frac{1}{v_2} - \frac{1}{v_{\text{min}}} \right) (x, p) dx$$

$$T_4(p, c) = \int_{-\sqrt{c+p_1}}^{p_2} \left( \frac{1}{v_1} - \frac{1}{v_{\text{min}}} \right) (x, p) dx + \int_{p_2}^{\sqrt{c+p_1}} \left( \frac{1}{v_2} - \frac{1}{v_{\text{min}}} \right) (x, p) dx.$$

As in all previous situations we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p \in S_1$ and write $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$. In order to preserve the last diagram we consider a new coordinate $\tilde{p}_1 = p_1 + c_1(p)$, which generically is justified by transversality theorems. For $p \in S_2$, $p \in S_3$ and $p \in S_4$ the optimal averaged profit is the unique solution vanishing at the origin of equations

$$(2) \ c([T_1 - ch](p, c)) = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} (x^2 - (c + p_1)) a(x, p) dx$$

$$(3) \ c([T_1 - ch](p, c)) = \int_{-\sqrt{c+p_1}}^{\sqrt{c+p_1}} (x^2 - (c + p_1)) b(x, p) dx$$

$$(4) \ c([T_1 - ch](p, c)) = \int_{-\sqrt{c+p_1}}^{p_2} (x^2 - (c + p_1)) a(x, p) dx + \int_{p_2}^{\sqrt{c+p_1}} (x^2 - (c + p_1)) b(x, p) dx,$$

respectively, where $a$ and $b$ are smooth functions negative at the origin.

As it was seen in singularity 4, solutions of equations (2) and (3) take the form

$$(2) \ c_2(p) = p_1^{3/2} C(p) + p_1^2 D(p) \quad \text{and} \quad (3) \ c_3(p) = p_1^{3/2} A(p) + p_1^2 B(p),$$

where all functions are smooth and positive at the origin. Therefore, these solutions are valid for $p_2 \geq p_1^{1/2} M(p, p_1^{1/2})$ and $p_2 \leq -p_1^{1/2} N(p, p_1^{1/2})$, respectively, where $M$ and $N$ are smooth functions with $M(0) = N(0) = 1$. Writing

$$M(p, p_1^{1/2}) = M_1(p) + p_1^{1/2} M_2(p) \quad \text{and} \quad N(p, p_1^{1/2}) = N_1(p) + p_1^{1/2} N_2(p),$$

for some smooth functions $M_1$, $M_2$, $N_1$ and $N_2$, where $M_2$ and $N_2$ have the same sign at the origin and $M_1(0) = N_1(0) = 1$, and choosing a new coordinate $\tilde{p}_2 = p_2 + \frac{1}{2} p_1 (N_2 - M_2)(p)$ we get that

$$S_2 = \{ p_2 \geq p_1 Q(p) + p_1^{1/2} M(p) \} \quad \text{and} \quad S_3 = \{ p_2 \leq -(p_1 Q(p) + p_1^{1/2} N(p)) \}.$$
for some new smooth functions $M$, $N$ and $Q$. Function $Q$ is eliminated when we divide the conditions that define $S_2$ and $S_3$ by $Q$ and when we consider $\tilde{p}_2 = (1/Q)(p)p_2$. As it was done in the study of singularity 4, functions $C$ and $D$ are removed choosing $\tilde{p}_1 = (D/C)^2(p)p_1$. In this simplification, the conditions that define regions $S_2$ and $S_3$ are also preserved if we consider $\tilde{p}_2 = (D/C)^2(p)p_2$. The previous coordinate changes do not change anything related to the first equation and so the fourth equation is valid for

$$-(p_1 + p_1^{1/2} N(p)) \leq p_2 \leq p_1 + p_1^{1/2} M(p).$$

Using the continuity of the optimal averaged profit (in this case it is difficult to use the differentiability) we obtain

$$p_1^{3/2} + p_1^2 + (p_2 - (p_1 + p_1^{1/2} M(p)))H(p_1^{3/2}, p),$$

as normal form for its solution, where $H$ is a smooth function.

**Figure 3.11: Singularities $11_{\pm}$**

**Singularities $12_{\pm}$.** Transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U = 2$

Suppose that the polydynamical system has exactly two admissible velocities, $v_1$ and $v_2$, and that the origin is a regular double point (for both extremal velocities) and that $f(0, 0) = 0$, which is a (relative minimum) of the profit density $f(\cdot, 0)$. The case of a minimum provides singularity $12_+$; the case of a maximum provides singularity $12_-$ and is proved in the same manner. As in the previous situation, after some coordinate changes we write the family of profit densities and the Maxwell set as $x^2 - p_1$ and $x = p_2$, respectively. However, in this situation the change of velocities is different as we can see in the following diagram.
Note that at points (except the origin) on the boundary of situation 1 the optimal averaged profit has a singularity 4 and on the boundary of situation 4 the profit has a singularity 3.

Consequently, around the origin the period of the $c$-level cycle is

$$T(p, c) = \begin{cases} T_1(p, c), & p \in S_1 = \{c + p_1 \leq 0\} \\ (T_1 + T_2)(p, c), & p \in S_2 = \{p_2 \geq \sqrt{c + p_1}\} \\ (T_1 + T_3)(p, c), & p \in S_3 = \{p_2 \leq -\sqrt{c + p_1}\} \\ (T_1 + T_4)(p, c), & p \in S_4 = \{|p_2| \leq \sqrt{c + p_1}\} \end{cases}$$

where $T_1$ is the period when, around the origin, we switch from $v_2$ to $v_1$ at $x = p_2$ and

$$T_2(p, c) = \int_{-\sqrt{c + p_1}}^{\sqrt{c + p_1}} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx$$

$$T_3(p, c) = \int_{-\sqrt{c + p_1}}^{\sqrt{c + p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx = -T_2(p, c)$$

$$T_4(p, c) = \int_{-\sqrt{c + p_1}}^{p_2} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx + \int_{p_2}^{\sqrt{c + p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (x, p) dx.$$

As in all previous situations we reduce by $R^+$-equivalence the optimal averaged profit to $c_1 = 0$ for $p \in S_1$ and write $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$. In order to preserve the last diagram we consider a new coordinate $\tilde{p}_1 = p_1 + c_1(p)$, which generically is justified by transversality theorems. For $p \in S_2$ and $p \in S_3$ the optimal averaged profit is the unique solution vanishing at the origin of equations (2-) and (2+), respectively, with

$$(2_{\pm}) \ c[(T_1 - ch)(p, c)] = \pm \int_{-\sqrt{c + p_1}}^{\sqrt{c + p_1}} (x^2 - (c + p_1))(x - p_2)a(x, p) dx,$$
where \( a \) is a smooth function negative at the origin. After some calculations, these equations are rewritten as
\[
(2_\pm) \, c \cdot \varphi(p, c) = \pm(c + p_1)^{3/2}[p_2 A(p, c) + (c + p_1)B(p, c)],
\]
for some smooth functions \( \varphi, A \) and \( B \), with \( \varphi \) and \( A \) both positive at the origin. Replacing, as before, for some smooth functions
\[\phi, S\]
boundary conditions of
\[D\]
for some smooth function
\[\therefore\]
\[p\]
equations are rewritten as
\[a\]
where
\[\therefore\]
\[\phi, S\]
functions
\[p\]
are now written as
\[\therefore\]
Choosing \( \tilde{c} \) is,
\[\therefore\]
as normal form for its solution, where
\[\therefore\]
\[H\]
is valid for
\[\therefore\]
\[\therefore\]
we obtain after some simplifications
\[\therefore\]
for some new smooth functions \( \varphi, A \) and \( B \), with \( \varphi \) and \( A \) both positive at the origin. Therefore, both solutions have the form
\[\therefore\]
for some smooth function \( D \) positive at the origin, with \( q = \pm p_1^{1/2} \). Then, one of the boundary conditions of \( S_2 \) and \( S_3 \) is
\[\therefore\]
respectively, for some smooth function \( M \) with \( M(0) = 1 \). Writing
\[\therefore\]
for some smooth functions \( M_1 \) and \( M_2 \) with \( M_1(0) = 1 \), they take the form
\[\therefore\]
Choosing \( \tilde{p}_2 = p_2 - p_1 M_2(p) \) and \( \tilde{p}_1 = M_1^2(p)p_1 \) we arrive to the form \( p_2 = \pm p_1^{1/2} \), that is,
\[\therefore\]
In this simplification we have considered new parameter coordinates and so solutions \( c_\pm \) are now written as
\[\therefore\]
where \( c_+ \) and \( c_- \) are valid for \( p \in S_2 \) and \( p \in S_3 \), respectively, for some new smooth functions \( A \) and \( D \) both positive at the origin. As usual, we write \( D(p, \pm p_1^{1/2}) = D_1(p) \pm p_1^{1/2}D_2(p) \), for some smooth functions \( D_1 \) and \( D_2 \) positive at the origin, and get
\[\therefore\]
Finally, we consider \( \tilde{p}_1 = (D_2/D_1)^2(p) \), \( \tilde{p}_2 = (D_2/D_1)(p) \) and \( \tilde{c} = (D_2^4/D_1^5)(p)c \) to remove functions \( D_1 \) and \( D_2 \) and preserve all regions \( S_i \). Therefore, the fourth equation is valid for \( -p_1^{1/2} \leq p_2 \leq p_1^{1/2} \). Using the differentiability of the optimal averaged profit we obtain
\[\therefore\]
as normal form for its solution, where \( H \) is a smooth function.
3.2.1.2 Singularities 13-15

Here we present a qualitative study for the last three singularities of Theorem 3.10 due to their complexity.

**Singularity 13.** Switching at a regular triple point and \#U = 3

Suppose that the polydynamical system has exactly three admissible velocities, \( v_1 \), \( v_2 \) and \( v_3 \), that the origin is a regular triple point (for both extremal velocities) and that \( f(0,0) = 0 \), which is a regular value of the profit density \( f(\cdot,0) \).

We can choose the same normal forms that were chosen in singularity 10 but now we obtain a completely different diagram of the change of velocities around the origin as we can see in the figure on the right. Note that at points (except the origin) on the boundary between situations 1 and 8 and between situations 4 and 5 the optimal averaged profit has a singularity 2 and on all the other boundaries this profit has a singularity 3.
3.2. Singularities of the optimal averaged profit for level cycles

So, in this singularity we have that the parameter plane is divided in eight different regions. Inside each region the optimal averaged profit is smooth and at points on the boundaries there are two types of differentiability of this profit: on the boundary $p_1 = 0$ it is of class $C^1$ and on all the other boundaries it is of class $C^2$.

We are able to present a normal form for this singularity, although it has a complicated expression. Around the origin, the period of the $c$-level cycle is

$$T(p, c) = \begin{cases} 
(\tilde{T} + T_1)(p, c), & p \in S_{1^+} = \{c + p_2 - p_1 \leq 0, p_1 \leq 0\} \\
(\tilde{T} + T_2)(p, c), & p \in S_{2^+} = \{c + p_2 - p_1 \geq 0, c + p_2 \leq 0\} \\
(\tilde{T} + T_3)(p, c), & p \in S_{3^+} = \{c + p_2 - \alpha(p) \leq 0, c + p_2 \geq 0\} \\
(\tilde{T} + T_4)(p, c), & p \in S_{4^+} = \{c + p_2 - \alpha(p) \geq 0, p_1 \leq 0\} \\
(\tilde{T} - T_1)(p, c), & p \in S_{1^-} = \{-c + p_2 - p_1 \leq 0, -p_1 \leq 0\} \\
(\tilde{T} - T_2)(p, c), & p \in S_{2^-} = \{-c + p_2 - p_1 \geq 0, -(c + p_2) \leq 0\} \\
(\tilde{T} - T_3)(p, c), & p \in S_{3^-} = \{-c + p_2 - \alpha(p) \leq 0, -(c + p_2) \geq 0\} \\
(\tilde{T} - T_4)(p, c), & p \in S_{4^-} = \{-c + p_2 - \alpha(p) \geq 0, -p_1 \leq 0\}
\end{cases}$$

where $\tilde{T}$ is the period when around the origin we just use the velocity $v_1$ and

$$T_1(p, c) = \int_{\frac{c + p_2}{\alpha(p)}}^{p_1} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx + \int_{p_1}^{\alpha(p)} \left(\frac{1}{v_3} - \frac{1}{v_1}\right) (x, p) \, dx$$

$$T_2(p, c) = \int_{\frac{c + p_2}{\alpha(p)}}^{\frac{c + p_2}{v_3}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx$$

$$T_3(p, c) = \int_{0}^{\frac{c + p_2}{v_3}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx + \int_{\frac{c + p_2}{v_3}}^{\frac{c + p_2}{\alpha(p)}} \left(\frac{1}{v_3} - \frac{1}{v_1}\right) (x, p) \, dx$$

$$T_4(p, c) = \int_{0}^{\frac{c + p_2}{v_3}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (x, p) \, dx.$$

As in all previous situations we write $\tilde{P}(p, c) = c^2 h(p, c)$, for some smooth function $h$ and, in order to preserve the last diagram, we consider a new coordinate $\tilde{p}_2 = p_2 + c_1(p)$, which generically is justified by transversality theorems. For $p \in S_{1\pm}$, $p \in S_{2\pm}$, $p \in S_{3\pm}$ and $p \in S_{4\pm}$ the optimal averaged profit is the unique solution vanishing at the origin of
Level cycles

for some smooth function \( C \), where all the functions are smooth, \( \tilde{\varphi} \), are of the form

\[
\varphi(x) = \begin{cases} 
\pm \alpha(p) & \text{if } x < c_p \\
0 & \text{if } x = c_p \\
\pm \alpha(p) & \text{if } x > c_p
\end{cases}
\]

where all functions are smooth and positive at the origin. Then, after some calculations we write these equations respectively, where \( \varphi = \tilde{T} - ch \) is a smooth function positive at the origin and \( a \) and \( b \) are smooth functions negative at the origin. After some calculations we write these equations as

\[
(1\pm) \ c\varphi(p, c) = \pm \int_{c_p}^{p_1} x(x - (c + p_2))a(x, p)dx \pm \int_{p_1}^{\alpha(p)} (x - \alpha(p))(x - (c + p_2))b(x, p)dx,
\]

\[
(2\pm) \ c\varphi(p, c) = \pm \int_{c_p}^{c_{p_2}} (x - \alpha(p))(x - (c + p_2))b(x, p)dx,
\]

\[
(3\pm) \ c\varphi(p, c) = \pm \int_{0}^{c_{p_2}} x(x - (c + p_2))a(x, p)dx \pm \int_{c_{p_2}}^{\alpha(p)} (x - \alpha(p))(x - (c + p_2))b(x, p)dx,
\]

\[
(4\pm) \ c\varphi(p, c) = \pm \int_{0}^{c_{p_2}} x(x - (c + p_2))a(x, p)dx
\]

respectively, where all functions are smooth and positive at the origin. Then,

\[
(1\pm) \ c\varphi(p, c) \pm \tilde{M}(p, c) \pm \tilde{N}(p, c) = \pm (\alpha(p) - (c + p_2))^2 \tilde{B}(p, c) \pm (c + p_2 - p_1)^2 \tilde{E}(p, c)
\]

\[
(2\pm) \ c\varphi(p, c) = \pm (\alpha(p) - (c + p_2))^2 \tilde{B}(p, c)
\]

\[
(3\pm) \ c\varphi(p, c) = \pm (\alpha(p) - (c + p_2))^2 \tilde{A}(p, c) \pm (c + p_2 - p_1)^2 \tilde{E}(p, c)
\]

\[
(4\pm) \ c\varphi(p, c) = \pm (c + p_2 - p_1)^3 \tilde{A}(p, c)
\]

where all functions are smooth and positive at the origin. Then, after some calculations we write these equations as

\[
(1\pm) \ c\varphi(p, c) \pm \tilde{M}(p, c) \pm \tilde{N}(p, c) = \pm (\alpha(p) - p_2)^2 \tilde{A}(p) \pm (p_2 - p_1)^2 \tilde{E}(p)
\]

\[
(2\pm) \ c\varphi(p, c) \pm \tilde{M}(p, c) = \pm (\alpha(p) - p_2)^2 \tilde{A}(p)
\]

\[
(3\pm) \ c\varphi(p, c) \pm \tilde{N}(p, c) = \pm (\alpha(p) - p_2)^2 \tilde{A}(p) \pm p_2 \tilde{B}(p)
\]

\[
(4\pm) \ c\varphi(p, c) = \pm p_2 \tilde{B}(p)
\]

where all the functions are smooth, \( \tilde{B}(p) = A(p, 0), \tilde{A}(p) = B(p, 0) \) and \( \tilde{E}(p) = E(p, 0) \). Function \( \tilde{B} \) is removed dividing all equations by itself and therefore all these equations are of the form

\[
c[\varphi + q\tilde{M} + r\tilde{N} + s\tilde{R}] = q(\alpha - p_2)^2 \tilde{A} + rp_2^3 + s(p_2 - p_1)^3 \tilde{E},
\]

where \( q, r \) and \( s \) take values in the set \( \{-1, 0, 1\} \). This fact implies that all solutions take the form

\[
c(p) = [q(\alpha - p_2)^2 \tilde{A} + rp_2^3 + s(p_2 - p_1)^3 \tilde{E}] \cdot C(p, q, r, s),
\]

for some smooth function \( C \) positive at the origin. So, we obtain the following solutions

\[
(1\pm) \ c_{1\pm}(p) = \pm [(\alpha(p) - p_2)^2 \tilde{A}(p) + (p_2 - p_1)^3 \tilde{E}(p)] \cdot C(p, \pm 1, 0, \pm 1)
\]

\[
(2\pm) \ c_{2\pm}(p) = \pm [(\alpha(p) - p_2)^2 \tilde{A}(p)] \cdot C(p, \pm 1, 0, 0)
\]

\[
(3\pm) \ c_{3\pm}(p) = \pm [(\alpha(p) - p_2)^2 \tilde{A}(p) + p_2^3] \cdot C(p, \pm 1, \pm 1, 0)
\]

\[
(4\pm) \ c_{4\pm}(p) = \pm p_2^3 \cdot C(p, 0, \pm 1, 0).
\]
For these normal forms we have

\[ S_{1+} = \{ p_2 \leq M_1(p) p_1, \ p_1 \leq 0 \}, \]
\[ S_{2+} = \{ M_1(p) p_1 \leq p_2 \leq M_2(p) p_1^3, \ p_1 \leq 0 \}, \]
\[ S_{3+} = \{ M_2(p) p_1^3 \leq p_2 \leq -M_3(p) p_1, \ p_1 \leq 0 \}, \]
\[ S_{4+} = \{ p_2 \geq -M_3(p) p_1, \ p_1 \leq 0 \}, \]
\[ S_{1-} = \{ p_2 \geq M_4(p) p_1, \ p_1 \geq 0 \}, \]
\[ S_{2-} = \{ M_5(p) p_1^3 \leq p_2 \leq M_4(p) p_1, \ p_1 \geq 0 \}, \]
\[ S_{3-} = \{ -M_6(p) p_1 \leq p_2 \leq M_5(p) p_1^3, \ p_1 \geq 0 \}, \]
\[ S_{4-} = \{ p_2 \leq -M_6(p) p_1, \ p_1 \geq 0 \}, \]

where all the functions are smooth and positive at the origin, \( M_1(0) = M_4(0) = 1 \).

![Figure 3.13: Singularity 13](image)

**Singularity 14.** Passing through a tangent double point with second order tangency

Suppose that the origin is a tangent double point with second order tangency inside the domain where one of the extremal velocities is used. Admit that \( v_1 \) and \( v_2 \) are the respective velocities that coincide at this point.

In this situation there are three conditions, namely, \( v_1 - v_2 = (v_1 - v_2)_x = (v_1 - v_2)_{xx} = 0 \) at the origin. Besides, \( (v_1 - v_2)_{xxx}(0) \neq 0 \) and then Mather Division Theorem implies that around the origin equation \( v_1 - v_2 = 0 \) is equivalent to \( x^3 + r_1(p)x^2 + r_2(p)x + r_3(p) = 0 \), where all \( r_i \) are smooth functions vanishing at the origin. Transversality theorems imply that generically

\[
\begin{vmatrix}
  r_{3,p1} & r_{3,p2} \\
  r_{2,p1} & r_{2,p2}
\end{vmatrix}
(0) \neq 0.
\]

Then, choosing new coordinates \( \tilde{x} = x + \frac{r_2}{4}(p), \tilde{p}_1 = r_2(p) - \frac{r_3^2}{4}(p) \) and \( \tilde{p}_2 = r_3(p) + \frac{2r_3^3}{27}(p) - \frac{r_1 r_2}{3}(p) \) we reduce, near the origin, the set where the extremal velocity used is not smooth to the form \( x^3 + p_1 x + p_2 = 0 \). In order to simplify our calculations, we consider a new
change of coordinates so that last equation takes the form \( x^3 - 3p_1 x - 2p_2 = 0 \). Without loss of generality we will assume that the velocity \( v_1 \) is used when \( x^3 - 3p_1 x - 2p_2 \leq 0 \).

It is well known from *Calculus* [24] that the discriminant of this equation is \( \Delta = p_1^3 - p_2^2 \) and that:

1) if \( \Delta < 0 \), this equation has a unique solution \( x_1 = (p_2 + \sqrt{-\Delta})^{1/3} + (p_2 - \sqrt{-\Delta})^{1/3} \); besides, it is easy to see that the sign of this solution coincides with the sign of \( p_2 \).

2) if \( \Delta = 0 \), this equation has solutions \( x_1 = 2p_2^{1/3} \) and \( x_2 = -p_2^{1/3} \) (with multiplicity 2); in this case it is also easy to see the relation between the signs of these solutions and of \( p_2 \).

3) if \( \Delta > 0 \), the equation has three solutions \( x_1, x_2 \) and \( x_3 \), which are expressed by Cardano’s Formula

\[
x_k = w^{k-1} \left( p_2 + i \sqrt{\Delta} \right)^{1/3} + \bar{w}^{k-1} \left( p_2 - i \sqrt{\Delta} \right)^{1/3}, \quad k = 1, 2, 3
\]

where \( w \) is a complex cubic root of the unity.

![Figure 3.14: Examples of graphics of the function \( x^3 - 3p_1 x - 2p_2 \).](image)

Consequently, around the origin the period of a \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
T_1(p, c) + T_2(p), & \Delta \leq 0 \\
T_1(p, c) + T_2(p) + T_3(p), & \Delta \geq 0
\end{cases}
\]

where \( T_1 \) is the period when, near the origin, we switch from \( v_1 \) to \( v_2 \) at \( x = 0 \) and

\[
T_2(p) = \int_0^{x_1} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx, \quad T_3(p) = \int_{x_2}^{x_3} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) (x, p) dx.
\]
3.2. Singularities of the optimal averaged profit for level cycles

Analogously we define the profit of a \( c \)-level cycle. As in all previous situations we write \( P_t(p, c) = c^2 h(p, c) \), for some smooth function \( h \). Besides,

\[
\frac{1}{v_1} - \frac{1}{v_2} = (x^3 - 3p_1x - 2p_2)a(x, p),
\]

for some smooth function \( a \) such that \( a(0) < 0 \) and \( (af)(0) > 0 \). For \( \Delta \leq 0 \) or \( \Delta \geq 0 \), the optimal averaged profit is the unique solution vanishing at the origin of equations

(1) \( c[(T_1 - ch)(p, c) + \int_0^{x_1} (x^3 - 3p_1x - 2p_2)a(x, p)] = \int_0^{x_1} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx, \)

(2) \( c[(T_1 - ch) + \int_0^{x_1} (x^3 - 3p_1x - 2p_2)a(x, p) + \int_{x_2}^{x_3} (x^3 - 3p_1x - 2p_2)a(x, p)dx] = \int_0^{x_1} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx + \int_{x_2}^{x_3} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx, \)

respectively. After some calculations it is possible to write each integral

\[
\int_0^{x_i} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx, \quad 1 \leq i \leq 3,
\]

as

\[
2x_i[p_2A(p) + p_1^2B(p)] + x_i^2[p_1A(p) + p_2B(p)] + p_2M(p),
\]

where all functions are smooth with \( A(0) < 0 \neq B(0) \) and \( M(0) = 0 \). Then,

\[
\int_0^{x_1} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx + \int_{x_2}^{x_3} (x^3 - 3p_1x - 2p_2)((af)(x, p))dx
\]

\[
= 2(x_1 - x_2 + x_3)(p_2A(p) + p_1^2B(p)) + (x_1^2 - x_2^2 + x_3^2)(p_1A(p) + p_2B(p)) + p_2M(p)
\]

\[
= 2(-2x_2)(p_2A(p) + p_1^2B(p)) + (-2x_2^2 + 6p_1)(p_1A(p) + p_2B(p)) + p_2M(p)
\]

Choosing \( \bar{p}_1 = (B/A)^2(p_1), \bar{p}_2 = -\text{sign}(B(0))(B/A)^3(p_2) \) and \( \bar{c} = (B^4/A^5)(c) \) as new coordinates, it is possible to remove functions \( A \) and \( B \) and, so, those equations take the form

(i) \( c[T_1 - ch(p, c) + 2y_i\hat{A}(p) + z_i\hat{B}(p) + p_2\hat{M}(p)] = 2y_i(p_2 + p_1^2) + z_i(p_1 + p_2) + p_2N(p), \)

where all functions are smooth and \( y_1 = x_1, \ z_1 = x_1^2, \ y_2 = -2x_2 \) and \( z_2 = -2x_2^2 + 6p_1 \). The last term of the right side is removed by \( R^+ \)-equivalence and, so, we are looking for solutions of an equation of the form

\[
c \cdot \psi(p, c, y, z) = 2y(p_2 + p_1^2) + z(p_1 + p_2).
\]

Hence, we conclude that all solutions are of the form

\[
c_i(p) = [2y_i(p_2 + p_1^2) + z_i(p_1 + p_2)] \cdot A(p, y_i, z_i)
\]
where \( y_1 = x_1, \ z_1 = x_1^2, \ y_2 = -2x_2 \) and \( z_2 = -2x_2^2 - 6p_1 \).

In this case, the normal form up to \( \Gamma \)-equivalence takes the form

\[
\begin{cases}
  c_1(p), & p_1^3 \leq p_2^2 \\
  c_2(p), & p_1^3 \geq p_2^2
\end{cases}, \quad c_i(p) = [2y_i(p_2 + p_1^2) + z_i(p_1 + p_2)]A(p, y_i, z_i)
\]

with \( y_1 = x_1, \ z_1 = x_1^2, \ y_2 = -2x_2, \ z_2 = -2x_2^2 - 6p_1 \), where \( x_1 \) is the root of the polynomial $x^3 - 3p_1x - 2p_2$ such that \( \text{sign} (x_1) = \text{sign} (p_2) \) and \( x_2 \) is the other root of this polynomial which is closer to the origin.

**Singularity 14.** Transition through a critical value which is not a minimum nor a maximum of the profit density provided by a point out of the Maxwell set.

Suppose that the origin is a critical point of the profit density \( f (\cdot, 0) \) providing the optimal level which is not a minimum nor a maximum. In this situation there are three conditions, namely, \( f = f_x = f_{xx} = 0 \) at the origin. Then, by genericity, \( v_{\min} - v_{\max} \) and \( f_{xxx} \) do not vanish at the origin. By classical results on Singularity Theory [6] we conclude that near the origin the family of profit densities is written as \( x^3 + \alpha(p)x + \beta(p) \), where \( \alpha \) and \( \beta \) are smooth functions vanishing at the origin. Transversality theorems justify that generically it is possible to choose new coordinates to write \( f \) as \( x^3 - 3p_1x - 2p_2 \). In this case, we have to switch from maximum velocity to minimum velocity when \( x^3 - 3p_1x - 2p_2 = c \). To simplify our calculations we consider a new coordinate \( c \) in order to write last equation as \( x^3 - 3p_1x - 2(p_2 + c) = 0 \). Remember that the discriminant of this equation is

\[
\Delta(p, c) = p_1^3 - (p_2 + c)^2
\]

and
3.2. Singularities of the optimal averaged profit for level cycles

1) if \( \Delta(p, c) < 0 \), this equation has a unique solution \( x_1 = (p_2 + c + \sqrt{-\Delta(p, c)})^{1/3} + (p_2 + c - \sqrt{-\Delta(p, c)})^{1/3} \); besides, it is easy to see that the sign of this solution coincides with the sign of \( p_2 + c \).

2) if \( \Delta(p, c) = 0 \), this equation has solutions \( x_1 = 2(p_2 + c)^{1/3} \) and \( x_2 = -(p_2 + c)^{1/3} \); in this case it is also easy to see the relation between the signs of these solutions and of \( p_2 + c \).

3) if \( \Delta(p, c) > 0 \), the equation has three solutions \( x_1, x_2 \) and \( x_3 \), which are expressed by Cardano’s Formula

\[
x_k(p, c) = w^{k-1} \left( p_2 + c + i\sqrt[3]{\Delta(p, c)} \right)^{1/3} + \bar{w}^{k-1} \left( p_2 + c - i\sqrt[3]{\Delta(p, c)} \right)^{1/3}, \quad k = 1, 2, 3
\]

where \( w \) is a complex cubic root of the unity.

As in the previous situation, around the origin, the period of a \( c \)-level cycle is

\[
T(p, c) = \begin{cases} (T_1 + T_2)(p, c), & \Delta(p, c) \leq 0 \\
(T_1 + T_2 + T_3)(p, c), & \Delta(p, c) > 0 \end{cases}
\]

where \( T_1 \) is the period when, near the origin, we switch from \( v_{\text{max}} \) to \( v_{\text{min}} \) at \( x = 0 \) and

\[
T_2(p, c) = \int_0^{x_1(p, c)} \left( \frac{1}{v_{\text{max}}} - \frac{1}{v_{\text{min}}} \right) (x, p) dx, \quad T_3(p, c) = \int_{x_2(p, c)}^{x_3(p, c)} \left( \frac{1}{v_{\text{max}}} - \frac{1}{v_{\text{min}}} \right) (x, p) dx.
\]

Analogously, we define the profit of a \( c \)-level cycle. As in all previous situations we write \( P_1(p, c) = c^2 h(p, c) \), for some smooth function \( h \). In order to preserve the previous normal forms we consider a new coordinate \( \bar{p}_2 = p_2 - c_1(p)/2 \) which generically is justified by transversality theorems. For \( \Delta(p, c) \leq 0 \) or \( \Delta(p, c) \geq 0 \), the optimal averaged profit is the unique solution vanishing at the origin of equations

1) \( c\varphi(p, c) = \int_0^{x_1(p, c)} (x^3 - 3p_1x - 2(p_2 + c))a(x, p) dx, \)

2) \( c\varphi(p, c) = \int_0^{x_1(p, c)} (x^3 - 3p_1x - 2(p_2 + c))a(x, p) dx + \int_{x_2(p, c)}^{x_3(p, c)} (x^3 - 3p_1x - 2(p_2 + c))a(x, p) dx, \)

respectively, where \( \varphi \) and \( a = 1/v_{\text{max}} - 1/v_{\text{min}} \) are smooth functions with \( \varphi(0) > 0 \) and \( a(0) < 0 \).

As it was said in the analysis of the previous singularity, it is possible to write each integral

\[
\int_0^{x_i(p, c)} (x^3 - 3p_1x - 2(p_2 + c))a(x, p) dx, \quad 1 \leq i \leq 3,
\]

as

\[
2x_i(p, c)[(p_2 + c)A(p, c) + p_1^2 B(p, c)] + x_i^2(p, c)[p_1 A(p, c) + (p_2 + c)B(p, c)] + (p_2 + c)M(p, c),
\]
where all functions are smooth with $A(0) > 0 \neq B(0)$ and $M(0) = 0$. Besides,

$$x_1(p,c) \int_0^{x_3(p,c)} (x^3 - 3p_1 x - 2(p_2 + c))a(x,p)dx + \int_{x_2(p,c)}^{x_3(p,c)} (x^3 - 3p_1 x - 2(p_2 + c))a(x,p)dx$$

$$= 2(-2x_2(p,c))(p_2 A(p,c) + p_1^2 B(p,c)) + (-2x_2^2(p,c) + 6p_1)(p_1 A(p,c) + p_2 B(p,c)) + (p_2 + c)M(p,c).$$

These results lead to the following equation

$$(i) c\varphi(p, c) = 2y[(p_2 + c)A(p,c) + p_1^2 B(p,c)] + z[p_1 A(p,c) + (p_2 + c)B(p,c)] + (p_2 + c)M(p,c),$$

(3.4)

where

1) $y = x_1(p,c)$, $z = x_1^2(p,c)$, if $\Delta(p,c) \leq 0$

2) $y = -2x_2(p,c)$, $z = -2x_2^2(p,c) + 6p_1$, if $\Delta(p,c) \geq 0$.

In this case it is very difficult to get an appropriate normal form for this singularity because $x_i$ are not differentiable when $c = 0$. Due to that, we just give a qualitative analysis of this solution:

I) In each region $\Delta(p,c) < 0$ and $\Delta(p,c) > 0$ the optimal cyclic profit is the unique ([9]) solution vanishing at the origin of equation (3.4); consequently, it is smooth at each point of these regions. Besides, when $p_2 = 0$ then $c_1 = 0$ is a solution of equation (3.4) in both regions. Then the solution in each region takes the form $p_2 \cdot \varphi(p)$, for some function $\varphi$.

II) The solution is continuous ([9]), because if $\Delta(p,c) = 0$ then $x_1(p,c) = -2x_2(p,c)$ and $x_1^2(p,c) = -2x_2^2(p,c) + 6p_1$.

III) More than continuous, we can prove that this solution is differentiable. In fact, in each region it is possible to estimate that $|\varphi(p)| < k|p|^{1+\varepsilon}$, for some constant $k$ and $\varepsilon > 0$.

IV) At the origin the solution is also differentiable but not of class $C^2$. In fact, it is very easy to find normal forms for this singularity in simple particular cases and, for example, it is easy to see that $p_2^{4/3} + p_2^{5/3}$ is a normal form when $p_1 = 0$. 

![Diagram](attachment:image.png)
3.2.2 Multi-point singularities

In this subsection we list all generic singularities of the optimal averaged profit for level cycles when, for a fixed value of the parameter, there are at least two points leading to a nonsmoothness situation listed on Theorem 3.10. The following result is useful to identify all possible generic situations.

Lemma 3.13: Consider a 2-parameter family of pairs of polydynamical systems and profit densities on the circle. Suppose that for a fixed value of the parameter there are exactly $N$ distinct points $Q_i$ ($1 \leq i \leq N$), each one leading to a codimension $c_i$ singularity of Tables 3.1 and 3.2, where $c_i = 2$ for singularities 1-4 and $c_i = 3$ for all the others. Generically,

$$\sum_{i=1}^{N} c_i \leq N + 2.$$  

Proof: Consider a point $(Q_1, \cdots, Q_N) \in (S^1 \times P)^{(N)}$ such that:

- each $Q_i$ leads to a codimension $c_i$ singularity of Tables 3.1 and 3.2,
- $\pi(Q_1) = \cdots = \pi(Q_N)$, where $\pi$ is the projection on the parameter space.

At such a point, a family of pairs of polydynamical systems and profit densities satisfies exactly $\mu$ independent equalities (among other conditions) with $\mu = \sum_{i=1}^{N} c_i + 2(N - 1)$. Consider the subset $Y$ of $(M \times P)^{(N)}$ consisting of the points at which just these equalities are satisfied. In the multijet bundle $J^4_N(S^1 \times P, \mathbb{R})$, the set $W$ of multijets of families at points of $Y$ is a codimension $\mu$ closed submanifold. Due to Multijet Transversality Theorem and the compactness of the circle, for a generic family $(V,f)$, the multijet extension $j^4_N(V,f)$ is transversal to $W$ and, therefore, $(j^5_N(V,f))^{-1}(W)$ is either empty or is a codimension $\mu$ submanifold of $(S^1 \times P)^{(N)}$. But $(j^5_N(V,f))^{-1}(W) = Y$ and $Y$ is a nonempty set. Therefore, generically this codimension can not be greater than the dimension $3N$ of $(S^1 \times P)^{(N)}$, and so, $\sum_{i=1}^{N} c_i \leq N + 2$. 

Theorem 3.14: Consider a 2-parameter family of pairs of polydynamical systems and profit densities on the circle. Generically, in the presence of exactly two of the situations 1–4 listed on Theorem 3.10 the germ of the optimal averaged profit for level cycles is, up to $\Gamma$-equivalence, the germ at the origin of one of the functions from the second column of Table 3.3.

Remark. Functions $A$, $B$ and $C$ are smooth functions of the parameter and are positive at the origin. In singularities 1, 7 and 8, $H \equiv \tilde{H} (p_1^{3/2}, p_2^{3/2}, p)$, $\tilde{H}$ smooth. In singularity 5,
Table 3.3:

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
<th>( p_1 \leq 0, p_2 \leq 0 )</th>
<th>( p_1 \geq 0, p_2 \leq 0 )</th>
<th>( p_1 \leq 0, p_2 \geq 0 )</th>
<th>( p_1 \geq 0, p_2 \geq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} ), ( p_{2/2} ), ( p_{3/2} ), ( p_{3/2} + p_{2/2} ), ( p_{3/2} + p_{2/2} + p_{3/2} + p_{3/2}/2H )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>2</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} ), ( p_{2/2} ), ( p_{1/2} + p_{2/2} ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2} )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>3</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} ), ( p_{1/2} + p_{2/2} ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2}A ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>4</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} ), ( p_{1/2} + p_{2/2} ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
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<tr>
<td>5</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
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<td>( p_{1/2} ), ( p_{1/2} + p_{2/2} ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>6</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} ), ( p_{1/2} + p_{2/2} ), ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
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<tr>
<td>7±</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} + p_{2/2}A ), ( p_{1/2} + p_{2/2}B )C, ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>7+</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2} + p_{2/2}A ), ( p_{1/2} + p_{2/2}B )C, ( p_{1/2} + p_{2/2} + p_{1/2}p_{2/2} + p_{1/2}p_{2/2}A )</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td>8±</td>
<td>0,</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
<tr>
<td></td>
<td>( p_{1/2}A ), ( p_{1/2}B )C, ( p_{1/2} + p_{2/2}A ), ( p_{1/2} + p_{2/2}B )C</td>
<td>( p_1 \leq 0, p_2 \leq 0 )</td>
<td>( p_1 \geq 0, p_2 \leq 0 )</td>
<td>( p_1 \leq 0, p_2 \geq 0 )</td>
<td>( p_1 \geq 0, p_2 \geq 0 )</td>
</tr>
</tbody>
</table>

\( H \equiv \tilde{H} \left( p_{3/2}^2, p \right) \), \( \tilde{H} \) smooth. In singularities 9 and 10, \( H \equiv \tilde{H} \left( p_{2/2}^3, p \right) \), \( \tilde{H} \) smooth. These singularities correspond to the following situations:
3.2. Singularities of the optimal averaged profit for level cycles

Table 3.3 (continued)

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
</tr>
</thead>
</table>
| 9±  | \[
<p>|     | \begin{cases}                                                       |
|     | 0, \quad p_1 \leq 0, p_2 \leq 0                                  |
|     | p_1^2 A, \quad p_1 \geq 0, p_2 \leq \mp p_1^2                   |
|     | p_2^{3/2} \pm p_2^2, \quad p_1 \leq 0, p_2 \geq 0                |
|     | p_1^2 A + (p_2 \pm p_1^2)H, \quad p_1 \geq 0, p_2 \geq \mp p_1^2 |</p>
<table>
<thead>
<tr>
<th></th>
<th>\end{cases}</th>
</tr>
</thead>
</table>
| 10± | \[
|     | \begin{cases}                                                       |
|     | 0, \quad p_1 \leq 0, p_2 \leq 0                                  |
|     | p_1^3 A, \quad p_1 \geq 0, p_2 \leq \mp p_1^3                    |
|     | p_2^{3/2} \pm p_2^2 B, \quad p_2 \geq 0, p_1 \leq -(p_2^{3/2} \pm p_2^2 B) |
|     | p_1^3 A + (p_2 \pm p_1^3)H, \quad p_2 \geq \mp p_1^3, p_1 \geq -(p_2^{3/2} \pm p_2^2 B) |
|     | \end{cases}                                                          |

1. Passing through two tangent double points (first order of tangency)
2. Passing through two regular triple points
3. Passing through a tangent double point and through a regular triple point
4. Switching at two regular double points
5. Passing through a tangent double point and switching at a regular double point
6. Passing through a regular triple point and switching at a regular double point
7. Transition through a local maximum and/or a local minimum of the profit density
8. Passing through a tangent double point and transition through a local minimum/maximum of the profit density
9. Passing through a regular triple point and transition through a local minimum/maximum of the profit density
10. Switching at a regular double point and transition through a local minimum/maximum of the profit density.

**Proof:** Consider a parameter value \( p_0 \) for which there are at least two points leading to a nonsmoothness situation listed on Theorem 3.10. Due to the previous lemma, there can not appear more than two points and all possible generic situations are those cited above. Let \((x_1, p_0)\) and \((x_2, p_0)\) be the points leading to one of singularities 1-4± of Table 3.1. Let \( c_0 \) be the respective optimal level. We select \( p_0 \) as the origin and, by \( R^+ \)-equivalence, we assume that \( c_0 = 0 \). All cases are proved using the same process and so we just consider the first case in detail.

1. Passing through two tangent double points (first order of tangency)

Suppose \((x_1, 0)\) and \((x_2, 0)\) are inside the domain where the extremal velocities are used and that both are tangent double points with first order tangency for these velocities. Let \( v_1 \) and \( v_2 \) be the respective velocities that coincide at \((x_1, 0)\) and \( v_3 \) and \( v_4 \) be the respective velocities that coincide at \((x_2, 0)\).
In this situation there are four conditions, namely, \( v_1 - v_2 = (v_1 - v_2)_x = 0 \) at \((x_1, 0)\) and \( v_3 - v_4 = (v_3 - v_4)_x = 0 \) at \((x_2, 0)\). Multijet Transversality Theorem implies that generically we can choose local coordinates \( y, z \) and \( p \) around \( x_1, x_2 \) and the origin, respectively, such that

\[
(v_1 - v_2)(y, p) = 0 \iff y^2 = p_1
\]

\[
(v_3 - v_4)(z, p) = 0 \iff z^2 = p_2.
\]

Without loss of generality we assume that the velocity used is

\[
\begin{align*}
  v_1(y, p), & \quad p_1 \leq y^2 \\
  v_2(y, p), & \quad p_1 \geq y^2
\end{align*}
\]

\[
\begin{align*}
  v_3(z, p), & \quad p_2 \leq z^2 \\
  v_4(z, p), & \quad p_2 \geq z^2
\end{align*}
\]

Consequently, around the origin, the period of the \( c \)-level cycle is

\[
T(p, c) = \begin{cases} 
  T_1(p, c), & p \in S_1 = \{p_1 \leq 0, p_2 \leq 0\} \\
  T_1(p, c) + T_2(p), & p \in S_2 = \{p_1 \geq 0, p_2 \leq 0\} \\
  T_1(p, c) + T_3(p), & p \in S_3 = \{p_1 \leq 0, p_2 \geq 0\} \\
  T_1(p, c) + (T_2 + T_3)(p), & p \in S_4 = \{p_1 \geq 0, p_2 \geq 0\}
\end{cases}
\]

where \( T_1 \) is the period when we just use velocity \( v_1 \) and velocity \( v_3 \) around \( x_1 = 0 \) and \( x_2 = 0 \), respectively, and

\[
T_2(p) = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) (y, p) dy \quad \text{and} \quad T_3(p) = \int_{-\sqrt{p_2}}^{\sqrt{p_2}} \left( \frac{1}{v_4} - \frac{1}{v_3} \right) (z, p) dz.
\]

As in all situations of the previous section we reduce by \( R^+ \)-equivalence the optimal averaged profit to \( c_1(p) = 0 \) for \( p \in S_1 \) and write \( P_1(p, c) = c^2 h(p, c) \), for some smooth function \( h \). For \( p \in S_2 \) and \( p \in S_3 \) the optimal averaged profit is the unique solution vanishing at the origin of

\[
(2) \ c[(T_1 - ch)(p, c)] = \int_{-\sqrt{p_1}}^{\sqrt{p_1}} (x^2 - p_1) [(af - ca)(x, p)] dx
\]

\[
(3) \ c[(T_1 - ch)(p, c)] = \int_{-\sqrt{p_2}}^{\sqrt{p_2}} (x^2 - p_2) [(bf - cb)(x, p)] dx,
\]

respectively, where \( a \) and \( b \) are smooth functions both positive at the origin. Using the study of singularity 1 in the previous section, the solutions of (2) and (3) can be written as

\[
c_2(p) = p_1^{3/2} A(p) + p_1^3 B(p) \quad \text{and} \quad c_3(p) = p_2^{3/2} C(p) + p_2^3 D(p),
\]

respectively, where all functions are smooth and positive at the origin. Choosing a new coordinate \( \tilde{c} \) such that \( c = \tilde{c} A(p) + \tilde{c}^2 B(p) \) we obtain

\[
c_2(p) = p_1^{3/2} \quad \text{and} \quad c_3(p) = p_2^{3/2} A(p) + p_2^3 B(p)
\]
for new smooth functions $A$ and $B$ positive at the origin. Now, functions $A$ and $B$ are removed doing $\tilde{p}_1 = p_1(B/A^2)^{2/3}(p)$, $\tilde{p}_2 = p_2(B/A)^{2/3}(p)$ and $\tilde{c} = (B/A^2)(p)c$. Finally, due to the continuity and differentiability of the optimal averaged profit we conclude that

$$
c_4(p) = p_1^{3/2} + p_2^{3/2} + p_2^3 + p_1^{3/2} p_2^{3/2} H(p_1^{3/2}, p_2^{3/2}, p)
$$

for some smooth function $H$ positive at the origin.

2. Passing through two regular triple points

Using the same process of the previous case and the study of singularity 2 of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

(1) $c_1(p) = 0$, $p \in S_1$
(2) $c_2(p) = p_1^2 A(p)$, $p \in S_2$
(3) $c_3(p) = p_2^2 B(p)$, $p \in S_3$

where the regions $S_i$ are exactly as in the previous case and $A$ and $B$ are smooth functions both positive at the origin. To simplify we choose $\tilde{p}_1 = p_1^{1/2} p_1$ and $\tilde{p}_2 = B^{1/2}(p)p_2$. Using the fact that at points on the boundary of $S_4$ the optimal averaged profit is of class $C^1$ we obtain for $p \in S_4$

$$
c_4(p) = p_1^2 + p_2^2 + p_1^2 p_2^2 H(p)
$$

for some smooth function $H$ positive at the origin. Choosing new coordinates $\tilde{p}_1 = p_1^{1/2} p_1$, $\tilde{p}_2 = p_2^{1/2}(p)p_2$ and $\tilde{c} = cH(p)$ we get singularity 2.

3. Passing through a tangent double point and a regular triple point

Using the previous two cases we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

(1) $c_1(p) = 0$, $p \in S_1$
(2) $c_2(p) = p_1^{3/2} A(p) + p_1^3 B(p)$, $p \in S_2$
(3) $c_3(p) = p_2^2 C(p)$, $p \in S_3$
where the regions \( S_i \) are exactly as in these cases and \( A, B \) and \( C \) are smooth functions both positive at the origin. These functions are removed doing \( \tilde{p}_1 = p_1(B/A)^{2/3} \), \( \tilde{p}_2 = p_2((BC)^{1/2}/A) + \tilde{c} = c(B/A^2)(p) \). Now, considering a new coordinate \( \tilde{p}_2 \) such that \( p_2 = \tilde{p}_2(1 + \tilde{p}_2^2)^{1/2} \) we get \( c_3(p) = p_2^2 + p_2^4 \), preserving all the other normal forms. After this coordinate change we consider \( \tilde{c} \) such that \( \tilde{c} = cB(p) \) and so we get singularity 3.

Using the differentiability of the optimal averaged profit we obtain for \( p \in S_4 \)

\[
\begin{align*}
\tilde{c}(p) &= p_1^{3/2} + p_2^2 + p_1^{3/2}p_2^2B(p) + p_1^2p_2^2A(p), & p \in S_4
\end{align*}
\]

for some smooth functions \( A \) and \( B \) both positive at the origin. Finally, function \( B \) is removed considering \( \tilde{p}_1 = p_1B^{2/3}(p) \), \( \tilde{p}_2 = p_2B^{1/2}(p) \) and \( \tilde{c} = cB(p) \) and so we get singularity 3.

![Figure 3.18: Sing. 3](image)

![Figure 3.19: Sing. 4](image)

4. Switching at two regular double points

Using the same process of the previous cases and the study of singularity 3 of the previous section we obtain that up to \( R^+ \)-equivalence the optimal averaged profit is, for the first three regions \( S_i \), written as

\[
\begin{align*}
(1) \ c_1(p) &= 0, & p \in S_1 \quad & \{c + p_1 \leq 0, c + p_2 \leq 0\} \\
(2) \ c_2(p) &= p_1^{3/2}A(p), & p \in S_2 \quad & \{c + p_1 \geq 0, c + p_2 \leq 0\} \\
(3) \ c_3(p) &= p_2^{3/2}B(p), & p \in S_3 \quad & \{c + p_1 \leq 0, c + p_2 \geq 0\}
\end{align*}
\]

where \( A \) and \( B \) are smooth functions both positive at the origin. We consider new coordinates \( \tilde{p}_1 = (A^{3/8}B^{1/8})(p)p_1 \), \( \tilde{p}_2 = (A^{1/8}B^{3/8})(p)p_2 \) and \( \tilde{c} = (A^{1/8}B^{3/8})(p)c \) to obtain

\[
\begin{align*}
(1) \ c_1(p) &= 0, & p_1 \leq 0, p_2 \leq 0 \\
(2) \ c_2(p) &= p_1^3, & p_1 \geq 0, p_2 \leq -p_1^3 \\
(3) \ c_3(p) &= p_2^3A(p), & p_2 \geq 0, p_1 \leq -p_2^3
\end{align*}
\]
for some new smooth function \( A \) positive at the origin. Finally, using the fact that at points (except the origin) on the boundary between \( S_2 \) and \( S_4 \) the optimal averaged profit is of class \( C^2 \) we obtain for \( p \in S_4 \)

\[
c_4(p) = p_1^3 + (p_2 + p_1^3)^3 B(p),
\]

for some new smooth function \( B \) positive at the origin and so we get singularity 4.

5. Passing through a tangent double point and switching at a regular double point

Using the same process of the previous cases and the study of singularities 1 and 3 of the previous section we obtain that up to \( R^+ \)-equivalence the optimal averaged profit is, for the first three regions \( S_i \), written as

\[
\begin{align*}
(1) & \quad c_1(p) = 0, \quad p \in S_1 = \{ p_1 \leq 0, c + p_2 \leq 0 \} \\
(2) & \quad c_2(p) = p_1^{3/2} A(p) + p_1^3 B(p), \quad p \in S_2 = \{ p_1 \geq 0, c + p_2 \leq 0 \} \\
(3) & \quad c_3(p) = p_2^3 C(p), \quad p \in S_3 = \{ p_1 \leq 0, c + p_2 \geq 0 \}
\end{align*}
\]

where \( A, B \) and \( C \) are smooth functions all positive at the origin. After an appropriate change of coordinates it is possible to get

\[
\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^{3/2} + p_1^3, \quad p_1 \geq 0, p_1^{3/2} + p_1^3 + p_2 A(p) \leq 0 \\
(3) & \quad c_3(p) = p_2^3, \quad p_1 \leq 0, p_2 \geq 0
\end{align*}
\]

for a new smooth function \( A \) positive at the origin. Choosing a new coordinate \( \tilde{p}_2 \) such that \( p_2 = \tilde{p}_2 (1 + \tilde{p}_2^{-1/3}) \) we get \( c_3(p) = p_2^3 + p_2^6 \), preserving all the other normal forms. After this coordinate change we consider \( \tilde{c} \) such that \( c = \tilde{c} + \tilde{c}^2 \) and so,

\[
\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^{3/2}, \quad p_1 \geq 0, p_1^{3/2} + p_1^3 + p_2 A(p) \leq 0 \\
(3) & \quad c_3(p) = p_2^3, \quad p_1 \leq 0, p_2 \geq 0
\end{align*}
\]

for some new smooth function \( A \) positive at the origin. Finally, we consider \( \tilde{p}_2 = A(p)p_2 \) and obtain

\[
\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^{3/2}, \quad p_1 \geq 0, p_2 \leq -(p_1^{3/2} + p_1^3) \\
(3) & \quad c_3(p) = p_2^3 A(p), \quad p_1 \leq 0, p_2 \geq 0
\end{align*}
\]

where \( A \) is a new smooth function positive at the origin. Finally, using the fact that at points (except the origin) on the boundary between \( S_2 \) and \( S_4 \) the optimal averaged profit is of class \( C^2 \) we obtain for \( p \in S_4 \)

\[
c_4(p) = p_1^{3/2} + (p_2 + p_1^{3/2} + p_1^3)^3 H(p_1^{3/2}, p),
\]

for some smooth function \( H \) positive at the origin.
6. Passing through a regular triple point and switching at a regular double point

Using the same process of the previous cases and the study of singularities 2 and 3 of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

$$(1) \ c_1(p) = 0, \quad p \in S_1 = \{p_1 \leq 0, c + p_2 \leq 0\}$$

$$(2) \ c_2(p) = p_1^2 A(p), \quad p \in S_2 = \{p_1 \geq 0, c + p_2 \leq 0\}$$

$$(3) \ c_3(p) = p_2^3 B(p), \quad p \in S_3 = \{p_1 \leq 0, c + p_2 \geq 0\}$$

where $A$ and $B$ are smooth functions both positive at the origin. Functions $A$ and $B$ are removed considering $\tilde{p}_1 = (A^{1/2}B^{1/4})(p)p_1$, $\tilde{p}_2 = B^{1/2}(p)p_2$ and $\tilde{c} = B^{1/2}(p)c$. Finally, using the fact that at points (except the origin) on the boundary between $S_2$ and $S_3$ the optimal averaged profit is of class $C^2$ we obtain for $p \in S_4$

$$c_4(p) = p_1^2 + (p_2 + p_1^2)^3 A(p),$$

for some smooth function $A$ positive at the origin.

7. Transition through a local minimum and/or a maximum of the profit density which are/is attained at two points

If $(x_1, 0)$ and $(x_2, 0)$ are the considered points then the signs

$$(1) + \quad (2) - \quad (3) + -$$

correspond to the following situations:

1. $(x_1, 0)$ and $(x_2, 0)$ are points of local minimum (for the profit density $f(\cdot, 0)$)
2. $(x_1, 0)$ and $(x_2, 0)$ are points of local maximum
3. $(x_1, 0)$ and $(x_2, 0)$ are points of local minimum and maximum, respectively.
3.2. Singularities of the optimal averaged profit for level cycles

All cases are treated in the same manner and so we just present the proof for the “+” case. The other two cases just need an adequate change of signs.

Using the same process of the previous cases and the study of singularities $4_\pm$ of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

1. \( c_1(p) = 0, \quad p \in S_1 = \{ p_1 + c \leq 0, \ p_2 + c \leq 0 \} \)
2. \( c_2(p) = p_1^{3/2}A_1(p) + p_1^2A_2(p), \quad p \in S_2 = \{ p_1 + c \geq 0, \ p_2 + c \leq 0 \} \)
3. \( c_3(p) = p_2^{3/2}B_1(p) + p_2^2B_2(p), \quad p \in S_3 = \{ p_1 + c \leq 0, \ p_2 + c \geq 0 \} \)

where all functions are smooth and positive at the origin. Considering new coordinates \( \tilde{p}_1 = (B_1^{4/5}A_1^{6/5})(p_1), \ \tilde{p}_2 = (B_1^{6/5}A_1^{4/5})(p_2) \) and \( \tilde{c} = (B_1^{6/5}A_1^{4/5})(c) \) we obtain

1. \( \tilde{c}(p) = 0, \quad \tilde{p}_1 \leq 0, \ \tilde{p}_2 \leq 0 \)
2. \( \tilde{c}(p) = p_1^{3/2} + p_1^2A(p), \quad \tilde{p}_1 \geq 0, \ \tilde{p}_2 \leq -(p_1^{3/2} + p_1^2A(p)) \)
3. \( \tilde{c}(p) = (p_2^{3/2} + p_2^2B(p))C(p), \quad \tilde{p}_2 \geq 0, \ \tilde{p}_1 \leq -(p_2^{3/2} + p_2^2B(p)) \)

for some new smooth functions $A$, $B$ and $C$ positive at the origin. Finally, using the continuity of the optimal averaged profit we get for $p \in S_4$

\[
\tilde{c}_4(p) = p_1^{3/2} + p_1^2A(p) + [p_2 + p_1^{3/2} + p_1^2A(p)]H(p_1^{3/2}, p_2^{3/2}, p),
\]

for some smooth function $H$ positive at the origin.

![Figure 3.22: Sing. 7+](image1)

![Figure 3.23: Sing. 7+-](image2)

![Figure 3.24: Sing. 7-](image3)

**8_±.** Passing through a tangent double point and transition through a local minimum/maximum of the profit density

Singularities $8_+$ and $8_-$ correspond to the minimum and maximum cases, respectively.

Using the same process of the previous cases and the study of singularities $1_\pm$ and $4_\pm$ of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

1. \( c_1(p) = 0, \quad p \in S_{1\pm} = \{ p_1 \leq 0, p_2 \pm c \leq 0 \} \)
2. \( c_2(p) = p_1^{3/2}A_1(p) + p_1^3A_2(p), \quad p \in S_{2\pm} = \{ p_1 \geq 0, p_2 \pm c \leq 0 \} \)
3. \( c_3(p) = p_2^{3/2}B_1(p) \pm p_2^2B_2(p), \quad p \in S_{3\pm} = \{ p_1 \leq 0, p_2 \pm c \geq 0 \} \)
where all functions are smooth and positive at the origin. Considering new coordinates $\tilde{p}_1 = (A_2/A_1)^{2/3} p_1$, $\tilde{p}_2 = (A_2/A_1^2)(p/p_2)$ and, after that, $\tilde{c}$ such that $c = (\tilde{c} + \tilde{c}^2)(A_1^2/A_2)(p)$ we obtain

\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^{3/2}, \quad p_1 \geq 0, p_2 \leq \mp(p_1^{3/2} + p_1^3) \\
(3) & \quad c_3(p) = p_2^{3/2}A(p) \pm p_2^2B(p), \quad p_1 \leq 0, p_2 \geq 0
\end{align*}

for new smooth functions $A$ and $B$ positive at the origin. Choosing $\tilde{c} = c/A(p)$ we obtain

\begin{align*}
& c_1(p) = p_1^{3/2}A(p) \quad \text{and} \quad c_2(p) = p_2^{3/2} \pm p_2^2B(p)
\end{align*}

for new smooth functions $A$ and $B$ positive at the origin, preserving all regions $S_i$. Finally, due to the continuity of the optimal averaged profit we conclude that for $p \in S_4$

\begin{align*}
& c_4(p) = p_1^{3/2}A(p) + (p_2 \pm (p_1^{3/2} + p_1^3))H(p_1^{3/2}, p_2^{3/2}, p)
\end{align*}

for some smooth function $H$ positive at the origin.

---

**9\_\pm**. Passing through a regular triple point and transition through a local minimum/maximum of the profit density

Singularities 9\_\pm\correspond to the minimum and maximum cases, respectively.

Using the same process of the previous cases and the study of singularities 2 and 4\_\pm of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

\begin{align*}
(1) & \quad c_1(p) = 0, \quad p \in S_{1\pm} \\
(2) & \quad c_2(p) = p_1^2A(p), \quad p \in S_{2\pm} \\
(3) & \quad c_3(p) = p_2^{3/2}B(p) \pm p_2^2C(p), \quad p \in S_{3\pm}
\end{align*}
3.2. Singularities of the optimal averaged profit for level cycles

where all functions are smooth and positive at the origin and $S_i$ are as in the previous case. Considering $\tilde{p}_1 = (A^{1/2}C/B)(p)$, $\tilde{p}_2 = (C/B)^2(p)p_2$ and $\tilde{c} = (C^3/B^4)(p)c$ we obtain

\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^2 A, \quad p_1 \geq 0, p_2 \leq \mp p_1^2 \\
(3) & \quad c_3(p) = p_2^{3/2} \pm p_2^2, \quad p_1 \leq 0, p_2 \geq 0,
\end{align*}

for a new smooth function $A$ positive at the origin. Finally, due to the continuity of the optimal averaged profit we conclude that for $p \in S_4$

$$c_4(p) = p_1^2 A + (p_2 \pm p_1^2)H(p_2^{3/2}, p)$$

for some smooth function $H$ positive at the origin.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3.27}
\caption{Sing. 9+}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3.28}
\caption{Sing. 9-}
\end{figure}

$10_\pm$. Switching at a regular double point and transition through a local minimum/maximum of the profit density

Singularities $10_+$ and $10_-$ correspond to the minimum and maximum cases, respectively. Using the same process of the previous cases and the study of singularities 3 and 4$_\pm$ of the previous section we obtain that up to $R^+$-equivalence the optimal averaged profit is, for the first three regions $S_i$, written as

\begin{align*}
(1) & \quad c_1(p) = 0, \quad p \in S_{1\pm} = \{c + p_1 \leq 0, p_2 \pm c \leq 0\} \\
(2) & \quad c_2(p) = p_1^3 A(p), \quad p \in S_{2\pm} = \{c + p_1 \geq 0, p_2 \pm c \leq 0\} \\
(3) & \quad c_3(p) = p_2^{3/2} B(p) \pm p_2^2 C(p), \quad p \in S_{3\pm} = \{c + p_1 \leq 0, p_2 \pm c \geq 0\}
\end{align*}

where all functions are smooth and positive at the origin. Considering new coordinates $\tilde{p}_1 = (A^{3/7}B^{2/7})(p)p_1$, $\tilde{p}_2 = (A^{2/7}B^{6/7})(p)p_2$ and $\tilde{c} = (A^{3/7}B^{2/7})(p)c$ we obtain

\begin{align*}
(1) & \quad c_1(p) = 0, \quad p_1 \leq 0, p_2 \leq 0 \\
(2) & \quad c_2(p) = p_1^3 A(p), \quad p_1 \geq 0, p_2 \leq \mp p_1^3 \\
(3) & \quad c_3(p) = p_2^{3/2} \pm p_2^2 B(p), \quad p_2 \geq 0, p_1 \leq -(p_2^{3/2} \pm p_2^2 B(p)),
\end{align*}
for some new smooth functions $A$ and $B$ positive at the origin. Finally, due to the continuity of the optimal averaged profit we conclude that for $p \in S_4$

$$c_4(p) = p_1^3 A(p) + (p_2 \pm p_1^3) H(p_2^{3/2}, p)$$

for some smooth function $H$ positive at the origin.

\[\blacksquare\]
Chapter 4

Transition between strategies

The optimal averaged profit can always be provided by either a stationary strategy or a level cycle [12]. In the last two chapters we have studied all generic singularities of this profit when it is just allowed to choose admissible motions among a unique type of such strategies. Each optimal strategy was considered separately. In the present chapter we study the behaviour of the optimal averaged profit in neighborhoods of parameter values where there are both types of strategies providing the optimal averaged profit and so, where it is necessary to switch between optimal strategies to get the optimal averaged profit.

**Definition 4.1:** A parameter value is called a transition value if in any neighborhood of it the maximum averaged profit can not be provided by one and only one type of strategy, namely, either by a level cycle or by a stationary strategy.

The following two lemmas play an important role to understand all generic cases that have to be considered.

**Lemma 4.2 ([12]):** At a transition value the optimal averaged profit is provided by a stationary strategy.

**Proof:** Let $p_0$ be a transition value, $S$ be the stationary domain and, for every parameter value $p$, $S(p)$ be the set of all points $x$ such that $(x,p)$ belongs to $S$. Firstly, note that $S(p_0)$ is a non empty set because otherwise, for all $p$ near $p_0$, the set $S(p)$ would also be empty (because $S$ is closed) and so, the optimal profit would be provided by level cycles. Therefore, we can consider the value $A_s(p_0)$.

Suppose, now, that the optimal averaged profit at $p_0$ is provided by a $c_0$-level cycle. Then, the averaged profit $A(p_0,c_0)$ along level cycles is greater than $A_s(p_0)$. Therefore,
the averaged profit $A$ along level cycles is defined around $(p_0, c_0)$ because for every $p$ around $p_0$, the set $I(p)$ of cyclic values is given by

$$I(p) = \begin{cases} [A_s(p), \max_{x \in S^1} f(x, p)], & \text{if } S(p) \neq \emptyset \\ [\min_{x \in S^1} f(x, p), \max_{x \in S^1} f(x, p)], & \text{if } S(p) = \emptyset \end{cases}$$

and $A_s$ is upper semi-continuous. But, due to Theorem 3.1, $A$ is a continuous function and so $A(p, c) > A_s(p)$ for all $(p, c)$ near $(p_0, c_0)$ where the inequality is defined. Consequently, $p_0$ is not a transition value.

**Lemma 4.3 ([12])**: Let $p_0$ be a transition value. If the optimal averaged profit $A_s$ for stationary strategies is continuous at $p_0$ and is defined in a neighborhood of it then $A_s(p_0) = A_l(p_0)$, where $A_l(p_0)$ is the upper limit of the averaged profits provided by level cycles as $p \to p_0$.

**Proof**: Due to the previous lemma, $A_s(p_0) \geq A_l(p_0)$. Suppose that $A_s(p_0) > A_l(p_0)$ and that $A_s$ is continuous at $p_0$ and defined around it. Then, for all points $(p, c)$ where $A$ is defined with $p$ around $p_0$, we have $A_s(p) > A(p, c)$ because $A$ is a continuous function and $A_s$ is continuous at $p_0$. Consequently, $p_0$ is not a transition value.

### 4.1 Generic cases

In this section we obtain all generic cases that have to be studied to obtain all generic singularities of the optimal averaged profit at transition values.

**Lemma 4.4**: Generically, if $p_0$ is a transition value then the profit $A_s(p_0)$:

1. just can be attained at points of the stationary domain of the following types: $I^2$, $A_0^1$, $A_1^1$, $A_{0,0}^1$, $A_0^2$, $A_1^2$;
2. can not be attained at more than two points. Moreover, if it is attained at exactly two points then they just can be of type $I^2$ or of type $A_0^1$.

**Proof**: Due to Corollary 2.13, generically $A_s(p_0)$ just can be attained at points of type $I^2$, $A_0^1$, $A_1^1$, $A_{0,0}^1$, $A_0^2$, $A_{0,0}^2$, $A_1^2$, $A_{0,0}^1$, $A_0^3$, $A_1^1$, $A_{0,0}^1$, $A_{0,0}^2$, $A_0^2$ and $A_0^3$ provide codimension 2 singularities which are continuous at the origin defined in a neighborhood of it and so, due to Lemma 4.3, an independent condition has to be satisfied. For this reason they have to be excluded and the first part of the lemma is proved. The second part follows from Multijet Transversality Theorem.
Lemma 4.5: Generically, if \( p_0 \) is a transition value whose profit \( A_s(p_0) \) is attained at an interior point of the stationary domain then \( A_s(p_0) \) can not be the global maximum of the corresponding profit density.

**Proof:** Let \((x_0, p_0)\) be an interior point of the stationary domain providing the global maximum of the profit density \( f(\cdot, p_0) \). Lemma 4.4 implies that the derivative \( f_{xx} \) does not vanish at this point. Then, due to Implicit Function Theorem, equation \( f_x(x, p) = 0 \) has a unique solution \( x = X(p) \) around this point, where \( X \) is a smooth function with \( X(p_0) = x_0 \). Consequently, for every \( p \) around \( p_0 \),

\[
A_s(p) = f(X(p), p) = \max_{x \in S_1} f(x, p)
\]

and so, \( p_0 \) is not a transition value.

Proposition 4.6: Let \( p_0 \) be a parameter value. If the profit \( A_s(p_0) \) is attained at a point of type \( A_{11} \), \( A_{01} \) or \( A_{12} \) then generically \( p_0 \) is not a transition value.

**Proof:** We prove that in a generic case the inequality \( A_I(p_0) > A_s(p_0) \) holds in all these cases and so, \( p_0 \) is not a transition value due to Lemma 4.2. Lemma 4.4 implies that in all these cases the profit \( A_s(p_0) \) is attained at a unique point \((x_0, p_0)\) of the stationary domain.

**Situation 1:** \((x_0, p_0)\) is a point of type \( A_{12} \)

In this situation there are three conditions, namely, \( f = 0 \) and \( v = v_x = 0 \) at \((x_0, p_0)\), for some admissible velocity \( v \), and so, in a generic case, no other independent condition can be satisfied. In particular, due to Lemma 4.3, \( A_s \) can not be continuous and defined in a neighborhood of \((x_0, p_0)\) which implies that \( A_s \) has singularity 7+ or 8 of Table 2.5. Moreover, due to Multijet Transversality Theorem, generically there can not appear a point distinct of \((x_0, p_0)\) satisfying one of the situations listed on Theorem 3.10.

We can consider (Chapter 2) a fibred local coordinate system with origin at \((x_0, p_0)\) where

\[
v_{\min}(x, p) = (x^2 - p_1) \cdot V(x, p),
\]

for some smooth function \( V \) positive at the origin, and the family of profit densities \( f \) is, up to \( \mathcal{F}^+ \)-equivalence, one of the functions \( \pm x^2 + p_2 x \). In this coordinate system we consider a sufficiently small neighborhood \([-a, a] \times [-\varepsilon, \varepsilon] \times [2]^2 \) of the origin where these normal forms take place.

(1) The normal form for \( f \) is \( x^2 + p_2 x \)

Consider \( p = 0 \) and, to simplify, \( f(x) = x^2 \) and \( v_{\min}(x) = x^2 h(x) \), for some smooth
function $h$ positive at the origin. All positive values of the profit density are cyclic values. Then,

$$\lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} \frac{P(c) + \int_{-a}^{-\sqrt{-c}} \frac{1}{h(x)} dx + \int_{-\sqrt{-c}}^{a} \frac{x^2}{\max(x)} dx + \int_{a}^{\sqrt{-c}} \frac{1}{h(x)} dx}{T(c) + \int_{-a}^{-\sqrt{-c}} \frac{1}{x^2h(x)} dx + \int_{-\sqrt{-c}}^{a} \frac{1}{\max(x)} dx + \int_{a}^{\sqrt{-c}} \frac{1}{x^2h(x)} dx}$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the $c$-level cycle outside the neighborhood $[-a,a]$ of $x_0$. After some calculations we conclude that such limit vanishes and consequently, $A_l(0) \geq 0$. The vanishing of $A_l - A_s$ at the origin gives an excessive independent condition on the transition and so, it does not take place in a generic case. Therefore, because $A_s(0) = 0$, we have $A_l(0) > 0$ and $0$ is not a transition value.

(2) The normal form for $f$ is $-x^2 + p_2x$

Consider $p_2 = 0$ and, to simplify, $f(x, p_1) = -x^2$ and $v_{\min}(x, p_1) = (x^2 - p_1)h(x, p_1)$, for some smooth function $h$ positive at the origin. When $p_1 < 0$ the optimal averaged profit has to be provided by cyclic strategies and so in a neighborhood of $c = 0$ all values are cyclic. We fix a negative value of $c$ and we study the averaged profit $A(p_1, c)$ when $p_1 \to 0^-$. The averaged profit $A(p_1, c)$ is given by

$$\frac{P(p_1, c) + \int_{-a}^{-\sqrt{-c}} \frac{-x^2}{\max(x,p_1)} dx + \int_{-\sqrt{-c}}^{a} \frac{-x^2}{(x^2-p_1)h(x,p_1)} dx + \int_{a}^{\sqrt{-c}} \frac{-x^2}{\max(x,p_1)} dx}{T(p_1, c) + \int_{-a}^{-\sqrt{-c}} \frac{1}{\max(x,p_1)} dx + \int_{-\sqrt{-c}}^{a} \frac{1}{(x^2-p_1)h(x,p_1)} dx + \int_{a}^{\sqrt{-c}} \frac{1}{\max(x,p_1)} dx}$$

where $P(p_1, c)$ and $T(p_1, c)$ are defined as previously. Simple calculations imply that the limit of $A(p_1, c)$ when $p_1 \to 0^-$ and $c \to 0^-$ is equal to $0$ and so, $A_l(0) \geq 0$. The conclusion is the same of the previous case.

**Situation 2**: $(x_0, p_0)$ is a point of type $A_{0,0}^1$

In this situation there are three conditions, namely, $f$ = 0 and $v_1 = v_2 = 0$ at $(x_0, p_0)$, for some admissible velocities $v_1$ and $v_2$, and $A_s(p_0) = A_l(p_0)$. Due to Multijet Transversality Theorem, generically there can not appear a point distinct of $(x_0, p_0)$ satisfying one of the situations listed on Theorem 3.10.

We can consider (Chapter 2) a fibred local coordinate system with origin at $(x_0, p_0)$ where

$$v_{\min}(x, p) = x(x - p_1) \cdot V(x, p),$$
for some smooth function \( V \) positive at the origin, and the family of profit densities \( f \) is, up to \( \mathcal{F}^+ \)-equivalence, the function \( x \). In this coordinate system we consider a sufficiently small neighborhood \([-a, a] \times [-\epsilon, \epsilon]^2\) of the origin where these normal forms take place.

Consider \( p = 0 \) and, to simplify, \( f(x) = x \) and \( v_{\text{min}}(x) = x^2 h(x) \), for some smooth function \( h \) positive at the origin. All positive values of the profit density are cyclic values. Then,

\[
\lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} \frac{P(c) + \int_{c}^{a} \frac{1}{x^n(x)} \, dx}{T(c) + \int_{c}^{a} \frac{1}{x^n(x)} \, dx}
\]

where \( P(c) \) and \( T(c) \) are the profit and the period, respectively, of the \( c \)-level cycle outside \([c, a]\). After some calculations (L'Hôpital’s Rule) we conclude that such limit is equal to \( 0^+ \). So, \( A_t(0) > 0 \) and 0 is not a transition value.

**Situation 3:** \((x_0, p_0)\) is a point of type \( A_1 \)

In this situation there are two conditions, namely, \( v = v_x = 0 \) at \((x_0, p_0)\), for some admissible velocity \( v \). Using Multijet Transversality Theorem in a generic case there can appear exactly one point \((x_1, p_0)\) distinct of \((x_0, p_0)\) leading to one of the nonsmoothness situations 1-4 listed on Theorem 3.10. Hence, this situation is divided in five cases:

**Case 1:** There are no points \((x, p_0)\) leading to a nonsmoothness situation listed on Theorem 3.10.

**Case 2:** There is a point \((x_1, p_0)\) distinct from \((x_0, p_0)\) leading to situation 1 of Theorem 3.10. In this case there are two additional conditions: \( v_1 - v_2 = (v_1 - v_2)_x = 0 \) at \((x_1, p_0)\), for some admissible velocities \( v_1 \) and \( v_2 \).

**Case 3:** There is a point \((x_1, p_0)\) distinct from \((x_0, p_0)\) leading to situation 2 of Theorem 3.10. In this case there are two additional conditions: \( v_1 - v_2 = v_1 - v_3 = 0 \) at \((x_1, p_0)\), for some admissible velocities \( v_1 \), \( v_2 \) and \( v_3 \).

**Case 4:** There is a point \((x_1, p_0)\) distinct from \((x_0, p_0)\) leading to situation 3 of Theorem 3.10. In this case there are two additional conditions: \( f = 0 \) and \( v_1 - v_2 = 0 \) at \((x_1, p_0)\), for some admissible velocities \( v_1 \) and \( v_2 \).

**Case 5:** There is a point \((x_1, p_0)\) distinct from \((x_0, p_0)\) leading to situations \( 4 \pm \) of Theorem 3.10. In this case there are two additional conditions: \( f = f_x = 0 \) at \((x_1, p_0)\).

In cases 2-5, generically \((x_1, p_0)\) can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

We can consider (Chapter 2) a fibred local coordinate system with origin at \((x_0, p_0)\) where

\[
v_{\text{min}}(x, p) = (x^2 - p_1) \cdot V(x, p),
\]
for some smooth function $V$ positive at the origin, and the family of profit densities $f$ is, up to $F^+$-equivalence, the function $x$. In this coordinate system we consider a sufficiently small neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place. Consider $p = 0$ and, to simplify, $f(x) = x$ and $v_{\min}(x) = x^2 h(x)$, for some smooth function $h$ positive at the origin. All positive values of the profit density are cyclic values.

In case 1,

$$
\lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} \frac{P(c) + \int_{-a}^{c} \frac{x}{v_{\max}(x)} dx + \int_{-a}^{a} \frac{1}{x h(x)} dx}{T(c) + \int_{-a}^{c} \frac{1}{v_{\max}(x)} dx + \int_{-a}^{a} \frac{1}{x h(x)} dx}
$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the $c$-level cycle outside the neighborhood $[-a, a]$ of $x_0$. As in the previous situation, this limit is equal to $0^+$. So, $A_l(0) > 0$ and 0 is not a transition value.

In all the other cases we can apply transversality theorems and obtain in a generic case a fibred local coordinate system $(y, p)$ with origin at $(x_1, 0)$ using the normal forms that were obtained in the proof of Theorem 3.12 and that are presented in Figure 4.1.

**Figure 4.1: Normal forms around $(x_1, p_0)$.**

For these cases we consider, in these new coordinate systems, that these normal forms take place in the neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin, obtaining that

$$
A(c) = \frac{P(c) + \int_{-a}^{c} \frac{x}{v_{\max}(x)} dx + \int_{c}^{a} \frac{1}{x h(x)} dx + P_1(c)}{T(c) + \int_{-a}^{c} \frac{1}{v_{\max}(x)} dx + \int_{c}^{a} \frac{1}{x h(x)} dx + T_1(p)}
$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the $c$-level cycle outside the neighborhoods $[-a, a]$ of $x_0$ and $x_1$ and $P_1$ and $T_1$ vary from case to case. Using the study of singularities 1-4 done above we obtain $P_1$ and $T_1$ for each case:
4.1. Generic cases

- In case 2, for \( p = 0 \) we just use \( v_1 \) in a neighborhood of \( x_1 \) and so

\[
P_1(c) = \int_{-a}^{a} \frac{f}{v_1}(y)dy, \quad T_1(c) = \int_{-a}^{a} \frac{1}{v_1}(y)dy.
\]

- In case 3, for \( p = 0 \) we just have to switch from velocity \( v_1 \) to velocity \( v_2 \) at \( y = 0 \) and so

\[
P_1(c) = \int_{-a}^{0} \frac{f}{v_1}(y)dy + \int_{0}^{c} \frac{f}{v_2}(y)dy, \quad T_1(c) = \int_{-a}^{0} \frac{1}{v_1}(y)dy + \int_{0}^{c} \frac{1}{v_2}(y)dy.
\]

- In case 4, we assume that the maximum velocity is not smooth (the other case is similar). For \( p = 0 \), we just have to switch from velocity \( v_1 \) to velocity \( v_2 \) at \( y = 0 \) and from velocity \( v_2 \) to \( v_{\text{min}} \) at \( y = c \) and so

\[
P_1(c) = \int_{-a}^{0} \frac{y}{v_1(y)}dy + \int_{0}^{c} \frac{y}{v_2(y)}dy + \int_{c}^{a} \frac{y}{v_{\text{min}}(y)}dy,
\]

\[
T_1(c) = \int_{-a}^{0} \frac{1}{v_1(y)}dy + \int_{0}^{c} \frac{1}{v_2(y)}dy + \int_{c}^{a} \frac{1}{v_{\text{min}}(y)}dy.
\]

- In case 5,

\[
P_1(c) = \int_{-a}^{a} \frac{y^2}{v_{\text{min}}(y)}dy + \int_{-\sqrt{c}}^{\sqrt{c}} \left( \frac{y^2}{v_{\text{max}}(y)} - \frac{y^2}{v_{\text{min}}(y)} \right)dy
\]

\[
T_1(c) = \int_{-a}^{a} \frac{1}{v_{\text{min}}(y)}dy + \int_{-\sqrt{c}}^{\sqrt{c}} \left( \frac{1}{v_{\text{max}}(y)} - \frac{1}{v_{\text{min}}(y)} \right)dy
\]

and

\[
P_1(c) = \int_{-a}^{a} \frac{-y^2}{v_{\text{max}}(y)}dy, \quad T_1(c) = \int_{-a}^{a} \frac{1}{v_{\text{max}}(y)}dy
\]

for the minimum and maximum cases, respectively.

In cases 2, 3, 4 and 5— we have that \( P_1 \) and \( T_1 \) are smooth and in case 5+ we have that both \( P_1 \) and \( T_1 \) vanish when \( c \to 0^+ \). In fact, in this last case \( T_1 \) and \( P_1 \) are integrals of the form \( \int_{0}^{\sqrt{c}} H(y^2)dy \), with \( H \) smooth, and so, because \( M(z) = \int_{0}^{c} H(y^2)dy \) is an odd function we conclude that \( \int_{0}^{\sqrt{c}} H(y^2)dy = \sqrt{c}N(c) \), for some smooth function \( N \). Therefore, the conclusion in cases 2-5 is exactly the same of the first case.

**Conclusion:** Due to the lemmas presented in this section we conclude that at a transition value \( p_0 \) the profit \( A_s(p_0) \) just can be attained at points of three types: \( I^2 \), \( A_0^1 \) and \( A_0^2 \). Besides, for all these cases the condition \( A_s(p_0) = A_l(p_0) \) has to be satisfied, due to Lemma 4.3.
4.2 Singularities of the optimal averaged profit at transition values

**Theorem 4.7:** Consider a $k$-parameter family of pairs of polydynamical systems and profit densities on the circle. Generically, the germ of the optimal averaged profit at a transition value is, up to the equivalence pointed out in the third column, the germ at the origin of one of the functions from the second column of:

- **Table 4.1**, if $k = 1$,
- **Tables 4.1 and 4.2**, if $k = 2$.

**Table 4.1:**

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\max{0; p_1}$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>2</td>
<td>$\max{0; -\frac{p_1}{\ln p_1} (1 + H_1)}$</td>
<td>$R^+$</td>
</tr>
</tbody>
</table>

**Remark.** These singularities correspond to the following situations:

1. $A_s$ attained at exactly one point, type $I^2$
2. $A_s$ attained at exactly one point, type $A_{01}^1$
3. $A_s$ attained at exactly one point, type $I^2$, and existence of another point leading to singularity 1 of Table 3.1
4. $A_s$ attained at exactly one point, type $I^2$, and existence of another point leading to singularity 2 of Table 3.1
5. $A_s$ attained at exactly one point, type $I^2$, and existence of another point leading to singularity 3 of Table 3.1
6. $A_s$ attained at exactly one point, type $I^2$, and existence of another point leading to singularities $A_{01}^1$ of Table 3.1
7. $A_s$ attained at exactly one point, type $A_{01}^1$, and existence of another point leading to singularity 1 of Table 3.1
8. $A_s$ attained at exactly one point, type $A_{01}^1$, and existence of another point leading to singularity 2 of Table 3.1
9. $A_s$ attained at exactly one point, type $A_{01}^1$, and existence of another point leading to singularity 3 of Table 3.1
Table 4.2:

<table>
<thead>
<tr>
<th>N.</th>
<th>Sing.</th>
<th>Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\max{0; p_1; p_2^{3/2}}$</td>
<td>$\Gamma$</td>
</tr>
<tr>
<td>4</td>
<td>$\max{0; p_1; p_2^2; p_2 \geq 0}$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>5</td>
<td>$\max{0; p_1; p_2^3; p_2 \geq 0}$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>6±</td>
<td>$\max{0; p_1; p_2^{3/2} \pm p_2^2}$</td>
<td>$\Gamma$</td>
</tr>
<tr>
<td>7</td>
<td>$\max{0; c(p) = \begin{cases} -\frac{p_1}{\ln p_1} (1 + H_1), &amp; p_2 \leq 0 \ -\frac{p_1 + p_2^{3/2}}{\ln(p_1 + p_2^{3/2})} (1 + H_2), &amp; p_2 \geq 0 \end{cases} }$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>8</td>
<td>$\max{0; c(p) = \begin{cases} -\frac{p_1}{\ln p_1} (1 + H_1), &amp; p_2 \leq 0 \ -\frac{p_1 + p_2^2}{\ln(p_1 + p_2^2)} (1 + H_2), &amp; p_2 \geq 0 \end{cases} }$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>9</td>
<td>$\max{0; c(p) = \begin{cases} -\frac{p_1}{\ln p_1} (1 + H_1), &amp; p_2 + c \leq 0 \ -\frac{p_1 + p_2^3}{\ln(p_1 + p_2^3)} (1 + H_2), &amp; p_2 + c \geq 0 \end{cases} }$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>10±</td>
<td>$\max{0; c(p) = \begin{cases} -\frac{p_1}{\ln p_1} (1 + H_1), &amp; p_2 \pm c \leq 0 \ -\frac{p_1 + p_2^{3/2}}{\ln(p_1 + p_2^{3/2})} (1 + H_2), &amp; p_2 \pm c \geq 0 \end{cases} }$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>11</td>
<td>$\max{0; p_1; p_2}$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>12</td>
<td>$\max{0; p_2; -\frac{p_1}{\ln p_1} (1 + H_1)}$</td>
<td>$R^+$</td>
</tr>
<tr>
<td>13</td>
<td>$\max{</td>
<td>p_1</td>
</tr>
<tr>
<td>14</td>
<td>$\begin{cases} 0, &amp; p_1 \leq 0 \ p_1, &amp; p_1 \geq 0, p_2 \geq 0 \ p_1 - p_2^2, &amp; p_1 \geq p_2^2 B, p_2 \leq 0 \ c(p), &amp; p_1 &lt; p_2^2 B, p_1 \geq 0, p_2 \leq 0. \end{cases}$</td>
<td>$R^+$</td>
</tr>
</tbody>
</table>

where $c$ is the unique solution vanishing at the origin of equation (4.3)

10±. $A_4$ attained at exactly one point, type $A_0^1$, and existence of another point leading to singularities $4_\pm$ of Table 3.1

11. $A_4$ attained at exactly two points, both type $I^2$

12. $A_4$ attained at exactly two points, types $I^2$ and $A_0^1$

13. $A_4$ attained at exactly two points, both type $A_0^1$

14. $A_4$ attained at exactly one point, type $A_0^2$.

Singualrities of Table 4.1 are already known [12].
Besides,

- In singularities 2, 7-10± and 12 of these tables, \( H_1 = h_1(p, \frac{1}{\ln(p_1)}, \frac{\ln|\ln(p_1)|}{\ln(p_1)}) \), where \( h_1 \) is a smooth function with \( h_1(p, 0, 0) \equiv 0 \)

- In singularity 7, \( H_2 = h_2(p, p_2^{3/2}, \frac{1}{\ln(p_1 + p_2^{3/2})}, \frac{\ln|\ln(p_1 + p_2^{3/2})|}{\ln(p_1 + p_2^{3/2})}) \), where \( h_2 \) is a smooth function with \( h_2(p, p_2^{3/2}, 0, 0) \equiv 0 \)

- In singularity 8, \( H_2 = h_2(p, \frac{1}{\ln(p_1 + p_2^{3/2})}, \frac{\ln|\ln(p_1 + p_2^{3/2})|}{\ln(p_1 + p_2^{3/2})}) \), where \( h_2 \) is a smooth function with \( h_2(p, 0, 0) \equiv 0 \)

- In singularity 9, \( H_2 = h_2(p, \frac{1}{\ln(p_1 + p_2^{3/2})}, \frac{\ln|\ln(p_1 + p_2^{3/2})|}{\ln(p_1 + p_2^{3/2})}) \), where \( h_2 \) is a smooth function with \( h_2(p, 0, 0) \equiv 0 \)

- In singularity 10, \( H_2 = h_2(p, \frac{1}{\ln(p_1 + p_2^{3/2})}, \frac{\ln|\ln(p_1 + p_2^{3/2})|}{\ln(p_1 + p_2^{3/2})}) \), where \( h_2 \) is a continuous function with \( h_2(p, 0, 0) \equiv 0 \)

- In singularity 13, \( H_q = h_q(p, \frac{1}{\ln G(p,q)}, \frac{\ln|\ln G(p,q)|}{\ln G(p,q)}) \), where \( h_q \) is a smooth function with \( h_q(p, 0, 0) \equiv 0 \), and \( \gamma_1 \) are smooth functions of the parameter

- In singularity 14, \( B \) is a smooth function of the parameter with \( B(0) > 1 \).

**Proof:** Consider a transition value \( p_0 \).

**Situation 1:** Suppose that the profit \( A_s(p_0) \) is attained at a unique point \((x_0, p_0)\) of the stationary domain: a point of type \( f^2 \). We can consider (Chapter 2) a fibred local coordinate system with origin at \((x_0, p_0)\) where the family of profit densities is, up to \( F^+\)-equivalence, the function \(-x^2\). By \( R^+\)-equivalence \( A_s \equiv 0 \) around the origin. Due to Lemma 4.5, 0 is not the global maximum of the profit density \( f(P, p_0) \) and so, all positive values of the density are cyclic. Then, the averaged profit \( A \) along level cycles is defined for all \((p, c)\) with \( p \) sufficiently close to \( p_0 \) and \( c > 0 \).

In this situation there are two conditions, namely, \( f_\xi(x_0, p_0) = 0 \) and \( A_s(p_0) = A_1(p_0) \). Therefore, using Multijet Transversality Theorem, in a generic case there can appear exactly one point \((x_1, p_0)\) distinct from \((x_0, p_0)\) leading to one of the nonsmoothness situations 1-4 listed on Theorem 3.10. Hence, this situation is divided in exactly the same five cases listed in situation 3 of the previous section on page 103. In cases 2-5, generically \((x_1, p_0)\) can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

In case 1,

\[
A(p, c) = \frac{P_1}{T_1}(p, c)
\]
where $P_1$ and $T_1$ are the profit and the period, respectively, of the $c$-level cycle. Let us analyse the form of the averaged profit $A$ along level cycles in cases 2-5 and only after that we treat cases 1-5 all together.

In all the other cases we can apply transversality theorems and obtain in a generic case a fibred local coordinate system $(y,p)$ with origin at $(x_1,0)$ using the normal forms that were obtained in the proof of Theorem 3.12 and that are presented in Figure 4.1 (page 104). Using the study of singularities 1-4 done in the proof of that theorem we obtain the formulas of the averaged profit $A$ along level cycles. For example, in case 2

$$A(p,c) = \begin{cases} 
\frac{P_1(p,c)}{T_1(p,c)}, & p_2 \leq 0 \\
\frac{P_1(p,c) + P_2(p)}{T_1(p,c) + T_2(p)}, & p_2 \geq 0
\end{cases}$$

where $P_1$ and $T_1$ are the profit and period, respectively, when around $x_0$ we just use the maximum velocity and around $x_1$ we just use velocity $v_1$ and

$$P_2(p) = \int_{-\sqrt{p_2}}^{\sqrt{p_2}} \left(f \cdot \left(\frac{1}{v_2} - \frac{1}{v_1}\right)\right) (y,p) dx$$

and

$$T_2(p) = \int_{-\sqrt{p_2}}^{\sqrt{p_2}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) (y,p) dx.$$

Now, for all cases 1-5 we consider an extension of the $A$ function to all $(p,c)$ around the origin considering the previous expressions also for $c \leq 0$. Now, equation $c = \left(P_1/T_1\right)(p,c)$ has a unique solution $c = c_1(p)$, where $c_1$ is a smooth function defined around the origin with $c_1(0) = 0$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is positive.

Subtracting this function from the family of profit densities we obtain $A_s = -c_1$ and so the optimal averaged profit is

$$\max\{-c_1(p), A_l(p)\},$$

where $A_l$ is the optimal averaged profit for level cycles. As it was seen in the proof of Theorem 3.12, it is possible to obtain, for the fixed coordinate systems, the following normal forms for $A_l$:

- In case 2, $A_l(p) = \max\{0; p_2^{3/2}A_1(p) + p_2^3A_2(p), p_2 \geq 0\}$
- In case 3, $A_l(p) = \max\{0; p_2^2A_3(p), p_2 \geq 0\}$
- In case 4, $A_l(p) = \max\{0; p_2^3A_4(p), p_2 \geq 0\}$
- In case 5, $A_l(p) = \max\{0; p_2^{3/2}A_5(p) \pm p_2^2A_6(p), p_2 \geq 0\}$

where all $A_i$ are smooth functions all positive at the origin.
Transversality theorems justify that generically $c_{1,p_1}(0) \neq 0$ (in case 1 we can assume it without loss of generality) and so we choose a new coordinate $\tilde{p}_1 = -c_1(p)$. Now it is easy to choose new coordinates on the parameter space for each case to get the following normal forms:

- In case 1, $\max\{0; p_1\}$
- In case 2, $\max\{0; p_1; p_2^{3/2}\}$
- In case 3, $\max\{0; p_1; p_2^2, p_2 \geq 0\}$
- In case 4, $\max\{0; p_1; p_2^3, p_2 \geq 0\}$
- In case 5, $\max\{0; p_1; p_2^{3/2} \pm p_2^2\}$.

**Situation 2:** Suppose that the profit $A_s(p_0)$ is attained at a unique point $(x_0, p_0)$ of the stationary domain: a point of type $A_0^1$. We can consider (Chapter 2) a fibred local coordinate system with origin at $(x_0, p_0)$ where

$$v_{\text{min}}(x, p) = x \cdot V(x, p),$$

for some smooth function $V$ positive at the origin, and the family of profit densities $f$ is, up to $\mathcal{F}^+$-equivalence, the function $x$. By $R^+$-equivalence $A_s \equiv 0$ around the origin. In this coordinate system we consider a sufficiently small neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place. In this situation there are two conditions, namely, $v_{\text{min}}(x_0, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$. Therefore, using Multijet Transversality Theorem, in a generic case there can appear exactly one point $(x_1, p_0)$ distinct from $(x_0, p_0)$ leading to one of the nonsmoothness situations 1-4 listed on Theorem 3.10. Hence, this situation is divided in exactly the same five cases listed in situation 3 of the previous section on page 103. In cases 2-5, generically $(x_1, p_0)$ can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

In case 1, for $c \in (0, a)$,

$$P(p, c) = \tilde{P}(p, c) + \int_c^a h(x, p)dx$$

$$T(p, c) = T^*(p, c) + \int_c^a \frac{1}{x} h(x, p)dx = \tilde{T}(p, c) - \gamma(p) \ln c,$$

where $h = 1/V$ and $\gamma$ is a smooth function positive at the origin. Note that $P$ and $\tilde{T}$ are smooth near $(p, c) = (0, 0)$. Therefore, equation $c = A(p, c)$ can be written as

$$c = \frac{P(p, c)}{T(p, c) - \gamma(p) \ln c}.$$

Function $\gamma$ is easily removed considering new smooth functions $P$ and $\tilde{T}$. Transversality theorems justify that generically $P_{p_1}(0) \neq 0$ or $P_{p_2}(0) \neq 0$. Without loss of generality, we
assume the first case. Writing \( P(p, c) = P(p, 0) + c\tilde{P}(p, c) \), for some smooth function \( \tilde{P} \) and choosing \( \tilde{p}_1 = P(p, 0) \) as a new coordinate we get the equation:

\[
c[H(p, c) - \ln c] = p_1,
\]

where \( H = \tilde{T} - \tilde{P} \) is a smooth function. Now, because \( \lim_{c \to 0^+} c[H(p, c) - \ln c] = 0^+ \) we conclude that near the origin this equation has no solution for \( p_1 \leq 0 \). For \( p_1 > 0 \) it has a solution of the form \( c(p) = -\frac{p_1}{\ln p_1} A(p) \), for some function \( A \) positive at the origin [12]. Replacing this form in the equation we obtain

\[
A(p) \left[ \frac{1}{\ln p_1} H \left( p, -\frac{p_1}{\ln p_1} A(p) \right) - \left( 1 + \frac{\ln A(p)}{\ln p_1} - \frac{\ln |\ln p_1|}{\ln p_1} \right) \right] + 1 = 0. \tag{4.1}
\]

Choosing new coordinates \( z = A(p), r = 1/\ln p_1 \) and \( s = \ln |\ln p_1|/\ln p_1 \), Equation (4.1) can be written as \( F(p, r, s, z) = 0 \), where

\[
F(p, r, s, z) = z[rH(p, -p_1 rz) - (1 + r \ln z - s)] + 1
\]

is a smooth function around \((0, 0, 0, A(0))\). Due to the Implicit Function Theorem we conclude that, around the considered point, Equation (4.1) is equivalent to \( z = Z(p, r, s) \), for some smooth function \( Z \) with \( Z(p, 0, 0) \equiv 1 \). Then,

\[
c(p) = -\frac{p_1}{\ln p_1} Z \left( p, \frac{1}{\ln p_1}, \frac{\ln |\ln p_1|}{\ln p_1} \right) = -\frac{p_1}{\ln p_1} \left[ 1 + H \left( p, \frac{1}{\ln p_1}, \frac{\ln |\ln p_1|}{\ln p_1} \right) \right],
\]

where \( H \) is a smooth function with \( H(p, 0, 0) \equiv 0 \) and, because the optimal averaged profit is \( \max\{A_s(p); c(p), c > 0\} \), we obtain singularity 2.

In all the other cases (2-5) we can apply transversality theorems and obtain in a generic case a fibred local coordinate system \((y, p)\) with origin at \((x_1, p_0)\) using the normal forms that were obtained in the proof of Theorem 3.12 and that are presented in Figure 4.1 (page 104). Using the study of singularities 1-4 done in the proof of that theorem and the arguments of the previous case we conclude that equation \( c = A(p, c) \) takes the form

### Case 2:

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c), \quad p_2 \leq 0
\]

\[
c[T(p, c) + p_2^{3/2}\hat{A}(p) - \gamma(p) \ln c] = P(p, c) + p_2^{3/2}A(p), \quad p_2 \geq 0
\]

### Case 3:

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c), \quad p_2 \leq 0
\]

\[
c[T(p, c) + p_2\hat{A}(p) - \gamma(p) \ln c] = P(p, c) + p_2^2A(p), \quad p_2 \geq 0
\]

### Case 4:

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c), \quad c - p_2 \leq 0
\]

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c) + (c - p_2)^3B(p, c), \quad c - p_2 \geq 0
\]

### Case 5:

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c), \quad p_2 \pm c \leq 0
\]

\[
c[T(p, c) - \gamma(p) \ln c] = P(p, c) + (p_2 \pm c)^3B(p, c), \quad p_2 \pm c \geq 0
\]

where all functions are smooth and, at the origin, \( A, B \) and \( \gamma \) are positive and \( P \) vanishes.

After this we remove \( \gamma \) considering new smooth functions \( P, T, A, \hat{A} \) and write
these equations as

\[
\begin{align*}
\text{Case 2: } c[H_1(p, c) - \ln c] &= P(p, 0), \quad p_2 \leq 0 \\
&= P(p, 0) + p_2^{3/2}A(p), \quad p_2 \geq 0 \\
\text{Case 3: } c[H_1(p, c) - \ln c] &= P(p, 0), \quad p_2 \leq 0 \\
&= P(p, 0) + p_2^2A(p), \quad p_2 \geq 0 \\
\text{Case 4: } c[H_1(p, c) - \ln c] &= P(p, 0), \quad c - p_2 \leq 0 \\
&= P(p, 0) - p_2^3A(p), \quad c - p_2 \geq 0 \\
\text{Case 5: } c[H_1(p, c) - \ln c] &= P(p, 0), \quad p_2 \pm c \leq 0 \\
&= P(p, 0) + p_2^{3/2}A(p), \quad p_2 \pm c \geq 0
\end{align*}
\]

where \( B \) is a continuous function and all the other functions are smooth with \( A \) positive at the origin. Finally, transversality theorems justify that generically \( P_{p_1}(0) \neq 0 \) and so, for each case it is easy to choose new coordinates on the parameter space in such a way that equation \( c = A(p, c) \) takes the form

\[
\begin{align*}
\text{Case 2: } c[H_1(p, c) - \ln c] &= p_1, \quad p_2 \leq 0 \\
&= p_1 + p_2^{3/2}, \quad p_2 \geq 0 \\
\text{Case 3: } c[H_1(p, c) - \ln c] &= p_1, \quad p_2 \leq 0 \\
&= p_1 + p_2^2, \quad p_2 \geq 0 \\
\text{Case 4: } c[H_1(p, c) - \ln c] &= p_1, \quad p_2 + c \leq 0 \\
&= p_1 + p_2^3, \quad p_2 + c \geq 0 \\
\text{Case 5: } c[H_1(p, c) - \ln c] &= p_1, \quad p_2 \pm c \leq 0 \\
&= p_1 + p_2^{3/2}, \quad p_2 \pm c \geq 0
\end{align*}
\]

where \( B \) is a continuous function and all the other functions are smooth. Therefore, proceeding as in case 1 we obtain singularities 7-10.

**Situation 3:** Suppose that the profit \( A_s(p_0) \) is attained at exactly two points \((x_0, p_0)\) and \((x_1, p_0)\) of the stationary domain, both of type \( I^2 \). In this situation there are four conditions, namely, \( f_s(x_0, p_0) = 0 \), \( f_s(x_1, p_0) = 0 \), \( f(x_0, p_0) - f(x_1, p_0) = 0 \) and \( A_s(p_0) = A_t(p_0) \). Due to transversality theorems, in a generic case no other independent condition can be satisfied.

Due to the Implicit Function Theorem, there are smooth functions \( \gamma_0 \) and \( \gamma_1 \) defined around \( p_0 \) such that

\[
f_x(\gamma_i(p), p) = 0 \text{ and } \gamma_i(p_0) = x_i \quad i = 1, 2.
\]

Therefore, for every \( p \) around \( p_0 \) the profit \( A_s \) is given by

\[
\max \{ f(\gamma_0(p), p), f(\gamma_1(p), p) \}.
\]
4.2. Singularities of the optimal averaged profit at transition values

In a generic case, because no other independent condition can be satisfied, none of the situations listed on Theorem 3.10 occurs. So, the averaged profit $A$ along level cycles is a smooth function defined for all $(p, c)$ with $p$ sufficiently close to $p_0$ and $c > A_s(p_0)$ as

$$A(p, c) = \frac{P(p, c)}{T(p, c)},$$

where $P$ and $T$ are the profit and the period, respectively, of the $c$-level cycle. Note that around $x_0$ and $x_1$ it is the maximum velocity that is used. As in situation 1, we consider an extension of the $A$ function to all $(p, c)$ around the origin considering the previous expression also for $c \leq 0$. Now, equation $c = A(p, c)$ has a unique solution $c = C(p)$, where $C$ is a smooth function defined around $p_0$ with $C(p_0) = A_s(p_0)$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is greater than $A_s(p_0)$.

Therefore, for every $p$ around $p_0$, the optimal averaged profit is given by

$$\max\{f(\gamma_0(p), p), f(\gamma_1(p), p), C(p)\}$$

which is $R^+$-equivalent to

$$\max\{0, f(\gamma_1(p), p) - f(\gamma_0(p), p), C(p) - f(\gamma_0(p), p)\}.$$  

Using Mutijet Transversality Theorem we can choose $\tilde{p}_1 = f(\gamma_1(p), p) - f(\gamma_0(p), p)$ and $\tilde{p}_2 = C(p) - f(\gamma_0(p), p)$ as new coordinates and so, we obtain singularity 11.

**Situation 4:** Suppose that the profit $A_s(p_0)$ is attained at exactly two points $(x_0, p_0)$ and $(x_1, p_0)$ of the stationary domain, one of type $I^2$ and another of type $A_0^1$, respectively. In this situation there are four conditions, namely, $f(x_0, p_0) - f(x_1, p_0) = 0$, $f_x(x_0, p_0) = 0$, $v(x_1, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$, for some admissible velocity $v$. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

We can consider (Chapter 2) fibred local coordinate systems $(x, p)$ and $(y, p)$ with origin at $(x_0, p_0)$ and $(x_1, p_0)$, respectively, where

$$v_{\min}(y, p) = y \cdot V(y, p),$$

for some smooth function $V$ positive at the origin, and the family of profit densities is written as $-x^2 + \alpha(p)$ and $y + \beta(p)$ around $(x_0, p_0)$ and $(x_1, p_0)$, respectively, for some smooth functions $\alpha$ and $\beta$ with $\alpha(0) = \beta(0)$. By $R^+$-equivalence the function $\beta$ can be removed and using Mutijet Transversality Theorem we can choose a new coordinate $p_1$ in such a way that $f(x, p) = -x^2 + p_1$. In the given coordinate systems, consider a neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the respective origins where the previous normal forms take place. Observe that, for every $p$ around the origin, $A_s(p) = \max\{0, p_1\}$. 
Due to transversality theorems, in a generic case no other independent condition can be satisfied and so, none of the situations listed on Theorem 3.10 occurs. So, the averaged profit $A$ along level cycles is defined for all $(p,c)$ with $p$ sufficiently close to 0 and $c > \max\{0,p_1\}$ as

$$A(p,c) = \frac{P(p,c) + \int_{-a}^{a} h(y,p)dy + \int_{-a}^{a} \frac{y}{V_{\max}(y,p)} dy}{T(p,c) + \int_{-a}^{a} \frac{1}{V_{\max}(y,p)} dy + \int_{-a}^{a} \frac{y}{2h(y,p)} dy + \int_{-a}^{a} \frac{1}{V_{\max}(x,p)} dx}$$

where $h = 1/V$ and $P(p,c)$ and $T(p,c)$ are the profit and the period, respectively, of the $c$-level cycle outside the neighborhoods $[-a,a]$ of $x_0$ and $x_1$. As in situation 1, we consider an extension of the $A$ function to all $(p,c)$ around the origin considering the previous expression also for $c \leq \max\{0,p_1\}$. Now, equation $c = A(p,c)$ has a unique solution $c = C(p)$, where $C$ is a smooth function defined around the origin with $C(0) = 0$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is positive. As it was seen in situation 2, this equation takes the form

$$c[H(p,c) - \ln c] = P(p,0)$$

for some smooth function $H$. The vanishing of the derivative $P_{p_2}$ at the origin gives an excessive independent condition on the transition and so, it does not take place in a generic case. Therefore, generically, we can consider $\tilde{p}_2 = P(p,0)$. As it was seen in the cited situation, this equation only has solution if $p_2 > 0$ which takes the form

$$C(p) = -\frac{p_2}{\ln p_2} \left[ 1 + H \left( p, \frac{1}{\ln p_2}, \frac{\ln |\ln p_2|}{\ln p_2} \right) \right],$$

for some smooth function $H$. The optimal averaged profit is, therefore, $\max\{0,p_1,C(p)\}$ and we obtain singularity 12.

**Situation 5:** Suppose that the profit $A_s(p_0)$ is attained at exactly two points $(x_0,p_0), (x_1,p_0)$ of the stationary domain, both of type $A_0^1$. In this situation there are four conditions, namely, $v_1(x_0,p_0) = 0$, $v_2(x_1,p_0) = 0$, $f(x_0,p_0) - f(x_1,p_0) = 0$ and $A_s(p_0) = A_t(p_0)$, for some admissible velocities $v_1$ and $v_2$. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

We can consider (Chapter 2) local coordinate systems $(x,p)$ and $(y,p)$ with origin at $(x_0,p_0)$ and $(x_1,p_0)$, respectively, where

$$v_{\min}(x,p) = x \cdot V_1(x,p) \quad \text{and} \quad v_{\min}(y,p) = y \cdot V_2(y,p)$$

for some smooth functions $V_1$ and $V_2$ positive at the corresponding origin, and the family of profit densities $f$ is written as $x + \alpha(p)$ and $y + \beta(p)$ around $(x_0,p_0)$ and $(x_1,p_0)$,
4.2. Singularities of the optimal averaged profit at transition values

In the given coordinate systems, consider a neighborhood $[-\delta, \delta] \times [-\varepsilon, \varepsilon]^2$ of the origin where the previous normal forms take place. Note that for every $p$ around the origin, $A_s(p) = |p_1|$. Due to transversality theorems, in a generic case no other independent condition can be satisfied and so, none of the situations listed on Theorem 3.10 occurs. Then, the averaged profit $A$ along level cycles is defined for all $(p, c)$ with $p$ sufficiently close to 0 and $c > |p_1|$ and equation $c = A(p, c)$ is written as

$$c = \frac{P_1(p, c) + \int_{c+p_1}^{a} \frac{x-p_1}{x} H_1(x, p) dx + \int_{c-p_1}^{a} \frac{y+p_1}{y} H_2(y, p) dy}{T_1(p, c) + \int_{c+p_1}^{a} \frac{1}{2} H_1(x, p) dx + \int_{c-p_1}^{a} \frac{1}{2} H_2(y, p) dy}$$

where all functions are smooth and, at the origin, $T_1, H_1$ and $H_2$ are positive and $P_1$ vanishes. This equation can be simplified to the form

$$c \cdot T_1(p, c) - P(p, c) = \gamma_1(p)(c + p_1) \ln(c + p_1) + \gamma_2(p)(c - p_1) \ln(c - p_1), \quad c > |p_1|,$$

where all functions are smooth and, at the origin, $T_1$, $\gamma_1$ and $\gamma_2$ are positive and $P$ vanishes. Now, $P(p, c) = P(p, 0) + c \tilde{P}(p, c)$ and generically the derivative $P_{p_2}$ does not vanish at the origin. Then, after a coordinate change, we obtain a new form for the last equation

$$c \cdot T(p, c) - p_2 = \gamma_1(p)(c + p_1) \ln(c + p_1) + \gamma_2(p)(c - p_1) \ln(c - p_1), \quad c > |p_1|, \quad (4.2)$$

where $T$ is a smooth function. The region where this equation has solution and the respective normal form can be obtained proceeding as in the previous situation 2. We obtain the following solution

$$c(p) = |p_1| - \frac{G(p, q)}{\ln G(p, q)} \left( 1/[(\gamma_1 + \gamma_2)(p) - q] + H_q \left( p, \frac{1}{\ln G(p, q)}, \frac{\ln |\ln G(p, q)|}{\ln G(p, q)} \right) \right)$$

valid for $G(p, q) > 0$ where

$$G(p, q) = \begin{cases} p_2 - |p_1| + q|2p_1| \ln |2p_1|, & p_1 \neq 0 \\ p_2, & p_1 = 0 \end{cases} \quad \text{and} \quad q = \begin{cases} \gamma_2(p), & p_1 < 0 \\ 0, & p_1 = 0 \\ \gamma_1(p), & p_1 > 0 \end{cases}$$

Therefore, the optimal profit is

$$A(p) = \max\{|p_1|; c(p), p_2 > 0 \wedge G(p, \gamma_1(p)) > 0 \wedge G(p, \gamma_2)(p) > 0\}.$$
Situation 6: Suppose that the profit $A_s(p_0)$ is attained at a unique point $(x_0, p_0)$ of the stationary domain, namely, a point of type $A_0^2$. In this situation, there are four conditions, namely, $f = f_x = 0$ and $v = 0$ at $(x_0, p_0)$, for some admissible velocity $v$, and $A_s(p_0) = A_l(p_0)$. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

We can consider (Chapter 2) a fibred local coordinate system with origin at this point where
\[ v_{\min}(x, p) = (x - p_2) \cdot V(x, p), \]
for some smooth function $V$ positive at the origin, and the family of profit densities $f$ is, up to $\mathcal{F}^+$-equivalence, the function $-x^2 + p_1$. Note that for every $p$ around the origin,
\[ A_s(p) = \begin{cases} p_1 - p_2^2, & p_2 \leq 0 \\ p_1, & p_2 \geq 0 \end{cases}. \]

In order to understand the optimal averaged profit provided by level cycles, we consider equation
\[ c = \frac{P_1}{T_1}(p, c), \]
where $P_1$ and $T_1$ are the profit and the period, respectively, when around the origin we just use the maximum velocity. Due to the Implicit Function Theorem, this equation has a unique solution $c = c_1(p)$, for some smooth function $c_1$ vanishing at the origin. Subtracting this function from the family of profit densities we reduce this solution to zero and the profit $P_1$ takes the form $P_1(p, c) = c^2 h(p, c)$, for some smooth function $h$.

Due to Multijet Transversality Theorem, we can consider $\tilde{p}_1 = p_1 - c_1(p)$. Observe that the switching to the minimum velocity just can give a better cyclic profit when $p_1 \geq 0$ and $p_2 \leq 0$. Therefore, the optimal averaged profit is
\[ A(p) = \begin{cases} 0, & p_1 \leq 0 \\ p_1, & p_1 \geq 0, p_2 \geq 0 \\ \max\{c(p), p_1 - p_2^2\}, & p_1 \geq 0, p_2 \leq 0 \end{cases}. \]
where $c$ is the unique solution vanishing at the origin of equation

$$
c^2 h(p, c) + \frac{\sqrt{p_1 - c}}{-\sqrt{p_1 - c}} \int_{-\sqrt{p_1 - c}}^{-\sqrt{p_1 - c}} (\frac{1}{v_{\min}} - \frac{1}{v_{\max}}) (x, p) dx
$$

Observe that

$$
\left( \frac{1}{v_{\min}} - \frac{1}{v_{\max}} \right) (x, p) = \frac{H(x, p)}{x - p_2},
$$

where $H$ is a smooth function positive at the origin. After some calculations it is possible to reduce this equation to the form

$$
c \varphi(p, c) - 2p_2 (p_1 - c)^{1/2} + (p_1 - c)^{3/2} \psi(p, c) = (p_1 - p_2^2 - c) \ln \left( \frac{(p_1 - c)^{1/2} - p_2}{-(p_1 - c)^{1/2} - p_2} \right), \quad (4.3)
$$

where $\varphi$ and $\psi$ are smooth functions with $\varphi(0) > 0$.

It is easy to see that this equation has no solution for $p_1 \geq p_2^2 B(p_2)$, for some smooth function $B$ with $B(0) > 1$. For $p_1 < p_2^2 B(p_2)$ its unique solution $c(p)$ vanishing at the origin is only defined for $c > p_1 - p_2^2$. Therefore, $c(p) > p_1 - p_2^2$ and we conclude that the optimal averaged profit is

$$
A(p) = \begin{cases} 
0, & p_1 \leq 0 \\
 p_1, & p_1 \geq 0, p_2 \geq 0 \\
 p_1 - p_2^2, & p_1 \geq p_2^2 B(p), p_2 \leq 0 \\
 c(p), & p_1 < p_2^2 B(p), p_1 \geq 0, p_2 \leq 0.
\end{cases}
$$

where $c$ is the unique solution vanishing at the origin of equation (4.3) and $B$ is a smooth function with $B(0) > 1$. 

![Figure 4.3: Singularity 14](image-url)
4. Transition between strategies
Conclusion

We classified all generic singularities of the optimal averaged profit for $k$-parameter families of pairs of polydynamical systems and profit densities on the circle when $k \leq 2$. Actually, the classification for $k = 1$ was already known [5], [9], [12], [21] and it consists of 10 different singularities. For $k = 2$ we have found more 58 different singularities, up to $\Gamma$-equivalence. I would like to leave a comment on these singularities:

1. **Stationary strategies singularities**: The process to obtain normal forms for this type of singularities is described on page 24. In this work we provide the necessary tools to find out the normal forms for higher dimensions of the parameter. So, we obtained normal forms for all these singularities.

2. **Cyclic strategies singularities**: the process to obtain normal forms for this type of singularities is much more complex than the previous one. In fact, for each situation we have to find normal forms for the Maxwell set and/or the family of profit densities and study all possible different manners of switching between the extremal velocities used. After that, for each subcase we have to write the so famous equation $c = A(p, c)$ in a simple form without integrals and, finally, we have to write a form for the corresponding solution. Relating these singularities I have to say that I am not completely satisfied with two facts which I tried to eliminate but with no success because time is always “running”:

- first, for singularity 15 we carried out all the steps except the last one, that is, although we could obtain for both subcases the respective form of equation $c = A(p, c)$ without integrals, we could not get the form for the corresponding solutions (we have, then, presented a qualitative study for this singularity);

- second, to write simple normal forms for the solutions of the first subcases, there was one solution (always left to the end) whose normal form was only possible to obtain using the differentiability of the optimal averaged profit (and in some situations only continuity was used).
I would like to remark that among this type of strategies we obtained a new kind of singularities which did not appear in the 1-dimensional parameter case, namely, \textit{multipoint singularities}.

3. \textbf{Transition singularities}: the process to obtain normal forms for this type of singularities depends from case to case, as we could see. There are situations for which the process is very simple and there are others for which the process is much more complex. Also in this study I have to point out that I am not completely satisfied because we could not find a form for the solution of equation $c = A(p, c)$ in the last singularity (14). Analogously as it was remarked for cyclic strategies, among this type of strategies there appeared singularities mixing both stationary and cyclic strategies, which did not happened in the simple case of a 1-dimensional parameter.

In these final words of this thesis I just would like to mention some other related works. As it was already said in \textit{Introduction}, V.I. Arnold introduced the problem that is treated in this thesis for the general case of control systems and obtained some singularities of the optimal averaged profit when the parameter is 1-dimensional [5]. This problem considered by Arnold is the basis of some other studies such as:

- [12], where the classification of all generic singularities of this problem was completed;
- ours, containing all generic singularities of the optimal averaged profit when the control system is a polydynamical system and when the parameter is 2-dimensional;
- [19], containing all generic singularities of the optimal averaged profit when the control space has a regular boundary;
- [14], on the study of the optimality of cyclic processes in the presence of a positive discount $\sigma$, that is, when in the expression of the averaged profit on the infinite horizon the functional under the integral sign contains the profit density multiplied by $e^{-\sigma t}$.

Note that many other studies can still arise from this problem such as, for example, consider situations presented in [12] and [19] with a higher dimension of the parameter (for which this thesis would be of good use) and consider the sphere as phase space of the control system.
References


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