Special Jordan Subspaces and Synchrony Subspaces in Coupled Cell Networks

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Abstract. Given a regular network (in which all cells have the same type and receive the same number of inputs and all arrows have the same type), we single out a class of special subspaces of eigenspaces and generalized eigenspaces, and we use these subspaces to study the synchrony phenomenon in the theory of coupled cell networks. To be more precise, we prove that the synchrony subspaces of a regular network are precisely the polydiagonals that are direct sums of these special subspaces. We also show that they play an important role in the lattice structure of all synchrony subspaces because every join-irreducible element of the lattice is the smallest synchrony subspace containing at least one of these special subspaces.

Key words. coupled cell networks, Jordan subspaces, synchrony, lattices

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1. Introduction. Networks are studied in many different areas of science, and examples are numerous and varied. For example, in chemistry networks can represent interactions between chemicals, in neuroscience networks can represent electrical and chemical synapses between neurons, in ecology networks can represent food webs, and in social sciences networks can represent communications between members of a group.

The interactions between the interconnected dynamical units and the influence of the network structure on the whole dynamics are of potential interest for scientists. For instance, perturbations in the internal dynamics of a unique dynamical unit can have a huge impact on the dynamics of the whole system. As an example, we mention the case of the Caribbean coral reefs. These coral reefs have been studied for many years, and they were thought to be resilient ecosystems because they were able to recover rapidly from many ecological and biological disturbances through the years. However, in the early 1980s, a sudden shift took the research community by surprise: the coral reefs collapsed to a small fraction of their original densities [13]. Underlying this ecological disturbance was a pathogen outbreak in the population of the sea urchin Diadema antillarum, which led to the collapse of this species (for instance, densities of this urchin on the Jamaican reefs crashed to 1% of their original level). The drastic reduction of this species caused a huge increase of the macro-algae and, because coral reefs and macro-algae compete in space, the reefs collapsed. Caribbean coral
reefs shifted to an algae-dominated state and showed an increasingly slow rate of recovery. This case shows that perturbations in the internal dynamics of a unique individual component (species) can strongly influence the dynamics of the whole system.

From the dynamical point of view in networks, it is also of interest to study when distinct individuals exhibit identical dynamics, at all instants of time, that are synchronized. For example, in the 17th century, the physicist Christiaan Huygens, inventor of the pendulum clock, recorded an interesting synchrony phenomenon: Huygens hung two clocks side by side on the same wooden beam, with opposite swings, and after a while he was surprised by the synchronization of the motion of each pendulum. Examples of synchrony also abound in nature and, undoubtedly, one of the most spectacular examples occurs when thousands of male fireflies gather in trees at night and flash in unison to attract females, providing a “silent, hypnotic concert” [16, p. 104]. However, not all examples of synchrony are associated with beautiful events. For instance, some brain diseases, such as epilepsy, result from an abnormal synchronization of a large number of neural populations [3]. Synchrony is an important phenomenon for networks in general, and in recent years it has been the subject of several research studies in many different areas, such as the Internet, spread of epidemics, food webs in ecosystems, neural circuits, gene transcription, and cellular signaling [5].

In the theory of coupled cell networks the concept of synchrony has always played a special role. Stewart, Golubitsky, and coworkers [15, 6] proved that it is possible to identify synchrony patterns in a network using solely the network architecture. It is surprising that we can identify synchrony in a network just by analyzing its structure, without knowing anything about the internal dynamics of each cell and the specific equations of the admissible coupled cell systems. Moreover, Stewart proved that the set of all synchrony subspaces of a general network is a complete lattice [14].

In this sense, it is important to study possible structures and properties of the set of all synchrony patterns in regular networks. We are led to important questions concerning synchrony in networks, such as: why does a network have a small or great number of synchrony subspaces? Where does the synchrony come from? What are the important features of the corresponding lattice of synchrony subspaces?

1.1. Regular coupled cell networks. In this subsection we briefly recall a few facts concerning the theory of (regular) coupled cell networks developed by Stewart, Golubitsky, and coworkers [15, 6].

A cell is a system of ordinary differential equations, and a coupled cell system is a finite collection of interacting cells. A coupled cell system can be associated with a network, a directed graph whose nodes represent cells and whose arrows represent couplings between cells. The general theory allows for loops and multiple arrows. All couplings of the same type between two cells are represented by a single arrow with the number of couplings attached to it, unless this number is equal to 1, in which case it is simply omitted. The general theory associates to each network a class of admissible vector fields, consistent with the structure of the network.

In this paper we restrict our attention to regular networks, that is, networks associated with coupled cell systems where all cells have the same differential equation (up to reordering of coordinates) and one kind of coupling. In this case, the state spaces of the cells are all
identical—say, a Euclidean space $\mathbb{R}^k$, with $k \geq 1$—and so, if the network has $n$ cells, then the total phase space is $(\mathbb{R}^k)^n$. The valency of a regular network is the number of arrows that input to each cell.

A polydiagonal is a subspace of the total phase space that is defined exclusively by the equalities of certain cell coordinates. The total phase space is polydiagonal, and the fully synchronous subspace is the polydiagonal defined by the equalities of all coordinates. A synchrony subspace is a polydiagonal that is flow invariant for every admissible vector field. Every regular network has at least two synchrony subspaces: the fully synchronous subspace and the total phase space. In this work, these are called the trivial synchrony subspaces.

Golubitsky, Stewart, and Török [6] proved that every coupled cell system associated with a network when restricted to a synchrony subspace corresponds to a coupled cell system associated with a smaller network, called the quotient network.

The adjacency matrix of a regular network is a square matrix $A = [a_{ij}]$, where $a_{ij}$ is the number of arrows that cell $i$ receives from cell $j$. Note that each row sum of the adjacency matrix equals the valency of the network.

Since every adjacency matrix is a multiple of a stochastic matrix, well-known results about stochastic matrices (see, for example, Meyer [12]) can be applied to adjacency matrices and guarantee that the following hold:

1. The valency $v$ is a semisimple eigenvalue (its algebraic and geometric multiplicities coincide) of the adjacency matrix, and, for every eigenvalue $\lambda$ of this matrix, $|\lambda| \leq v$.

2. $(1, \ldots, 1)$ is an eigenvector associated with the valency.

1.2. Synchrony subspaces. There are in the literature some well-known methods for finding synchrony subspaces. The first method is due to Golubitsky, Stewart, and Török [6] and is based on the following concept: an equivalence relation $\bowtie$ on the set $C$ of cells is balanced if any pair of cells in a $\bowtie$-equivalence class $c$ receive the same number of inputs from cells in the $\bowtie$-equivalence class $d$ for each $d$. For general networks, it is assumed that all cells in the same $\bowtie$-equivalence class are of the same type. Coloring all cells of the network so that two cells have the same color precisely when they are $\bowtie$-equivalent, $\bowtie$ is balanced if and only if cells with the same color receive equal number of inputs from cells of any given color. Choosing a total phase space $P$, consider the polydiagonal

$$\Delta_{\bowtie} = \{x \in P : x_c = x_d \text{ whenever } c \bowtie d \forall c, d \in C\}.$$ 

Theorem 4.3 of [6] states that $\Delta_{\bowtie}$ is a synchrony subspace if and only if $\bowtie$ is balanced. This combinatorial method for finding synchrony subspaces uses exclusively the network architecture.

This theorem also allows us to relate synchrony subspaces of regular networks to invariant subspaces of the corresponding adjacency matrix $A$ in the following way: a polydiagonal is a synchrony subspace if and only if it is $A$-invariant, assuming that each cell phase space is $\mathbb{R}$. Taking this relation into account, Proposition 2.3 of Aguiar et al. [2] provides another simple method for identifying, by a matrix computation on the adjacency matrix, all synchrony subspaces of a regular network. More recently, Kamei and Cock [10] generalized this method to general networks, providing an algorithm for determining, by a matrix computation on a symbolic adjacency matrix encoding distinct arrow types, all synchrony subspaces of a general
network and the corresponding lattice.

Algebraic methods for finding synchrony subspaces were also motivated by Theorem 4.3 of [6] and were based on the aforementioned relation between synchrony subspaces of regular networks and invariant subspaces of the corresponding adjacency matrix. This relation was studied in detail by Kamei for the special case of simple eigenvalues in [8], and this work was the main motivation of our paper. More recently, Aguiar and Dias [1] extended this work to general networks and provided an algorithm for constructing the lattice of all synchrony subspaces through a small set of synchrony subspaces. They also proved that the problem of obtaining the lattice of synchrony subspaces of a general network can be reduced to the problem of obtaining the lattice of synchrony subspaces of regular networks.

Our work offers a new algebraic way of finding synchrony subspaces. We prove that all synchrony subspaces of a general regular network can be obtained using the direct sum operation over a small set of Jordan subspaces—the special Jordan subspaces. Recall that in Aguiar and Dias [1] it is proved that all synchrony subspaces of a general network can be obtained using the sum operation over a small set of synchrony subspaces—the sum-irreducible synchrony subspaces. We emphasize that although our initial aim was to generalize the work of Kamei [8] to regular networks, our principal aim has become to demonstrate the close relationship between synchrony subspaces of regular networks and special Jordan subspaces of the corresponding adjacency matrices.

1.3. Jordan subspaces. In this subsection we briefly recall some basic concepts and results about Jordan subspaces that are extremely important in our work. For details about this subject, see, for example, section 6 of Lancaster and Tismenetsky [11].

Consider a linear transformation $A$ from $\mathbb{C}^n$ into $\mathbb{C}^n$ and an eigenvalue $\lambda$. For a positive integer $r$, the subspace $K^r_\lambda = \ker (A - \lambda I)^r$ is the generalized eigenspace of order $r$ (to $\lambda$). Since $A$ has finite order, there is a positive integer $p$ such that $K^1_\lambda \subset \cdots \subset K^p_\lambda = K^{p+1}_\lambda = \cdots$.

For such a $p$, the generalized eigenspace $K^p_\lambda$ is the generalized eigenspace to $\lambda$, and it is simply denoted by $G_\lambda$. An element $x$ in $K^p_\lambda \setminus K^{p-1}_\lambda$ is called a generalized eigenvector of order $r$ of $A$, considering $K^0_\lambda = \{0\}$. An ordinary eigenvector is a generalized eigenvector of order 1, and an eigenspace is a generalized eigenspace of order 1.

A sequence of nonzero vectors $\{x_1, \ldots, x_k\}$ is called a Jordan chain of length $k$ associated with the eigenvalue $\lambda$ when

$$(A - \lambda I)x_1 = 0, \quad (A - \lambda I)x_2 = x_1, \quad \ldots, \quad (A - \lambda I)x_k = x_{k-1}.$$ 

Any Jordan chain consists of linearly independent vectors [11, p. 230], and so, if it has length $k$, it spans a $k$-dimensional subspace. A Jordan subspace is a subspace spanned by the vectors of some Jordan chain.

1.4. Synchrony subspaces and Jordan subspaces. As mentioned above, Theorem 4.3 of [6] allows us to relate synchrony subspaces of regular networks to invariant subspaces of the corresponding adjacency matrix $A$, in the following way: a polydiagonal is a synchrony subspace if and only if it is $A$-invariant, assuming that each cell phase space is $\mathbb{R}$. 
In Aguiar and Dias [1] it is carefully explained that in the calculation of synchrony subspaces each cell phase space can be extended from $\mathbb{R}$ to $\mathbb{C}$. Basically and briefly (for more details, see [1]), this is due to the fact that a vector $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$ satisfies an equality of coordinates $x_i = x_j$ if and only if $\text{Re}(v_i) = \text{Re}(v_j)$ and $\text{Im}(v_i) = \text{Im}(v_j)$, $1 \leq i < j \leq n$.

On the other hand, given a linear transformation $A$ from $\mathbb{C}^n$ into $\mathbb{C}^n$, it is well known that a subspace is $A$-invariant if and only if it is a direct sum of Jordan subspaces [11]. Thus, we obtain the following result, which will play a special role throughout this work and which can be interpreted as a corollary of Theorem 4.3 of [6].

**Lemma 1.1.** Given a regular network, a polydiagonal is a synchrony subspace if and only if it is a direct sum of Jordan subspaces of the corresponding adjacency matrix, assuming that each cell phase space is $\mathbb{C}$.

In particular, if the adjacency matrix of a regular network has only simple eigenvalues, then all synchrony subspaces are polydiagonal and direct sums of eigenspaces. For example, the adjacency matrix of the regular network in Figure 1 has only simple eigenvalues, and the corresponding eigenspaces are

$$
G_2 = \{x_1 = x_2 = x_3 = x_4\},
G_0 = \{x_1 = x_4, x_2 = x_3, x_1 + x_2 = 0\},
G_1 = \{x_1 = x_3, x_1 + x_2 = 0, x_4 = 0\},
G_{-1} = \{x_1 = x_2 = x_3, 2x_1 + x_4 = 0\}.
$$

Calculating all possible direct sums of these eigenspaces and then selecting only those that are polydiagonal, we conclude that there are only four nontrivial synchrony subspaces:

1. $G_2 \oplus G_0 = \{x_1 = x_4, x_2 = x_3\}$,
2. $G_2 \oplus G_{-1} = \{x_1 = x_2 = x_3\}$,
3. $G_2 \oplus G_1 \oplus G_{-1} = \{x_1 = x_3\}$,
4. $G_2 \oplus G_0 \oplus G_{-1} = \{x_2 = x_3\}$.

Notice, for example, that $G_2 \oplus G_1 = \{x_1 = x_3, x_1 + x_2 - x_4 = 0\}$ is not polydiagonal, because it is defined by an equality that is not an equality of cell coordinates. As a consequence, it is not a synchrony subspace.

This case of simple eigenvalues is special because, for a regular network, all synchrony subspaces can be obtained simply by computing all possible sums (finite and direct) of such eigenspaces. We can use this fact to make some important observations about the lattices of
these networks. For example, when \( n \geq 4 \), we can guarantee the existence of some possible lattice structures such as pentagons. In lattice theory, the pentagon lattice is very popular and has many important properties, such as that it is one of the simplest examples of nondistributive lattices and is the smallest nonmodular lattice [7]. Kamei [8] proved that when \( n = 4 \), there is only one possible pentagon structure, which has two vertices as 2-dimensional synchrony subspaces and one vertex as a 3-dimensional synchrony subspace. When \( n \) increases, the number of possible pentagon structures also increases; for example, for \( n = 5 \) there are four possible pentagon structures, which are shown in Figure 2.

![Figure 2. All possible pentagon lattices for 5-cell regular networks whose adjacency matrices have only simple eigenvalues. Level \( L_i \) contains \( i \)-dimensional synchrony subspaces, with \( 1 \leq i \leq 5 \).](image)

The main result of this paper, Theorem 5.1, strengthens Lemma 1.1: given a regular network, we single out a class of special subspaces of eigenspaces and generalized eigenspaces and prove that a polydiagonal is a synchrony subspace if and only if it is a direct sum of Jordan subspaces in that class. All the results preceding this theorem are auxiliary to understanding it and show that the highlighted Jordan subspaces have many important algebraic properties.

1.5. Structure of the paper. In section 2 we motivate and introduce the key concept of this work, the special Jordan subspace. In section 3 we define special subspaces, which are subspaces that are very important to calculating special Jordan subspaces, as well as to understanding some of their important algebraic properties. In section 4 we prove Theorem 4.9, which is the main result about special Jordan subspaces: given an \( n \)-cell regular network, there is always a direct sum decomposition of \( \mathbb{C}^n \) into special Jordan subspaces. In section 5 we prove Theorem 5.1, which is the main result of this work about synchrony and shows the important role played by special Jordan subspaces in the synchrony phenomenon: given a regular network, a polydiagonal is a synchrony subspace if and only if it is a direct sum of special Jordan subspaces. We also describe a method for listing all synchrony subspaces. In section 5 we study lattices consisting of all synchrony subspaces, and we show that special Jordan subspaces play an important role in the structure of these lattices, being directly connected with the join-irreducible elements of such lattices; precisely, every join-irreducible element is the smallest synchrony subspace containing at least one special Jordan subspace.

2. Special Jordan subspaces. In this section we motivate and introduce the key concept of this work: special Jordan subspaces.

2.1. Motivation. Lemma 1.1 guarantees that all synchrony subspaces of a regular network can be written as a direct sum of Jordan subspaces of the corresponding adjacency matrix. Since synchrony subspaces are defined by equalities of coordinates, the knowledge of all synchrony subspaces can be obtained from the list of all Jordan subspaces satisfying at
least one equality of coordinates. Nevertheless, some observations in concrete examples made us believe that some Jordan subspaces were essential in this list and some others were not. Then, we conjectured the existence of a class of Jordan subspaces sufficient to generate all synchrony subspaces by direct sums.

**Example 2.1.** The regular network in Figure 3 has only five nontrivial synchrony subspaces, namely,

\[ S_1 = \{x_1 = x_2 = x_3, x_4 = x_5\}, \]
\[ S_2 = \{x_1 = x_4 = x_5, x_2 = x_3\}, \]
\[ S_3 = \{x_2 = x_3 = x_4 = x_5\}, \]
\[ S_4 = \{x_2 = x_3, x_4 = x_5\}, \]
\[ S_5 = \{x_1 = x_4\}. \]

Its adjacency matrix has four distinct eigenvalues, namely, 2, −1, and ±i, and the corresponding eigenspaces are

\[ G_2 = \{x_1 = x_2 = x_3 = x_4 = x_5\}, \]
\[ G_{-1} = \{x_2 = x_3, x_4 = x_5, x_1 + x_2 + x_4 = 0\}, \]
\[ G_{\pm i} = \{x_1 = x_4, 2x_1 + 2x_2 + x_3 = 0, \pm (2i)x_1 + x_3 = 0, 3x_1 + x_2 + x_5 = 0\}. \]

Notice that −1 has algebraic multiplicity 2 and geometric multiplicity 2.

Observing that

\[ S_i = G_2 \oplus W_i \quad \text{for } 1 \leq i \leq 3, \]
\[ S_4 = G_2 \oplus W_1 \oplus W_2, \]
\[ S_5 = G_2 \oplus W_2 \oplus G_i \oplus G_{-i}, \]

where \( W_1, W_2, \) and \( W_3 \) are the following Jordan subspaces in \( G_{-1} \),

\[ W_1 = \{x_1 = x_2 = x_3, x_4 = x_5, 2x_1 + x_4 = 0\}, \]
\[ W_2 = \{x_1 = x_4 = x_5, x_2 = x_3, 2x_1 + x_2 = 0\}, \]
\[ W_3 = \{x_2 = x_3 = x_4 = x_5, x_1 + 2x_2 = 0\}, \]

we understand that, among all Jordan subspaces in \( G_{-1} \), the subspaces \( W_1, W_2, \) and \( W_3 \) are essential to listing all nontrivial synchrony subspaces using direct sums.

**2.2. Definition.** Motivated by the above conjecture, we singled out the following class of Jordan subspaces.

**Definition 2.2.** Consider an \( n \)-cell regular network. A Jordan subspace \( W \) in a generalized eigenspace \( G \) of the corresponding adjacency matrix is special to the network when every
Jordan subspace $U$ in $G$ such that
\[ \dim U = \dim W \quad \text{and} \quad P(U) \subset P(W) \]
satisfies
\[ P(U) = P(W) \quad \text{or} \quad U = F, \]
where $F$ is the fully synchronous subspace and, for a subspace $V$ of $\mathbb{C}^n$, $P(V)$ denotes the smallest polydiagonal containing $V$, that is, the intersection of all polydiagonals containing $V$.

Thus, $W$ is a special Jordan subspace of the network when every Jordan subspace $U$ in $G$ having the same dimension and satisfying the same equalities of coordinates as $W$ either does not satisfy any additional equality of coordinates or is the fully synchronous subspace.

**Example 2.3.** In the previous example, applying this definition directly, $G_2$ and $G_{\pm i}$ are 1-dimensional special Jordan subspaces. $G_{-1}$ is a 2-dimensional generalized eigenspace of order 1, and thus all Jordan subspaces (and special Jordan subspaces) in $G_{-1}$ are 1-dimensional. In particular, $G_{-1}$ is not a Jordan subspace. All 1-dimensional subspaces of $G_{-1}$ satisfy at least two equalities of coordinates, namely, $x_2 = x_3$ and $x_4 = x_5$. Hence, because the fully synchronous subspace—the unique 1-dimensional subspace that satisfies four independent equalities of coordinates—is not a subspace of $G_{-1}$, all special Jordan subspaces in $G_{-1}$ satisfy three independent equalities of coordinates. Therefore, there are exactly three 1-dimensional subspaces in $G_{-1}$: $W_1$, $W_2$, and $W_3$. Then, there are exactly six special Jordan subspaces of this network. Notice that all of them are essential to listing all nontrivial synchrony subspaces of the network using direct sums.

We use this example to point out some important facts about special Jordan subspaces:

1. Not all Jordan subspaces are special Jordan subspaces. For instance, $U = \{x_1 = 0, x_2 = x_3, x_4 = x_5, x_2 + x_4 = 0\}$ is a 1-dimensional Jordan subspace in $G_{-1}$ that is not a special Jordan subspace because $W_3$ is also a 1-dimensional Jordan subspace in $G_{-1}$ and
\[ P(W_3) = \{x_2 = x_3 = x_4 = x_5\} \varsubsetneq P(U) = \{x_2 = x_3, x_4 = x_5\}. \]

2. The definition of special Jordan subspaces strongly depends on the corresponding eigenspace. Indeed, $G_{\pm i}$ and $W_2$ are 1-dimensional special Jordan subspaces and $P(W_2) \subset P(G_{\pm i})$. This fact is possible only because the subspaces belong to distinct eigenspaces.

3. The number of independent equalities of coordinates satisfied by all special Jordan subspaces with a prescribed dimension is not fixed, in the sense that two special Jordan subspaces with the same dimension may satisfy a different number of equalities of coordinates.

**Remark 2.4.** In a generalized eigenspace of order $r \geq 1$ there are $k$-dimensional Jordan subspaces for every $1 \leq k \leq r$ and so, applying the definition directly, there are special Jordan subspaces of all possible dimensions, 1 up to $k$.

### 3. Special subspaces.

In this section we define *special subspaces*, and we present important properties of these subspaces that will be used throughout this work. These subspaces are essential to identifying special Jordan subspaces.
3.1. Definition. Given a subspace $W$ of $\mathbb{C}^n$, we denote by $P(W)$ the smallest polydiagonal containing $W$, that is, the intersection of all polydiagonals containing $W$. Notice the following basic properties of this operator, for all subspaces $W$ and $V$ of $\mathbb{C}^n$:

1. $W \subset P(W)$,
2. $P(W \cap V) \subset P(W) \cap P(V)$,
3. $W \subset V \Rightarrow P(W) \subset P(V)$,
4. $W$ is polydiagonal $\iff P(W) = W$.

Definition 3.1. Consider a subspace $E$ of $\mathbb{C}^n$. A subspace $W$ of $E$ is special in $E$ when every subspace $U$ of $E$ such that $
 \dim U = \dim W \quad \text{and} \quad P(U) \subset P(W)$

satisfies the equality $P(U) = P(W)$.

Thus, $W$ is special in $E$ when every subspace $U$ of $E$ having the same dimension and satisfying the same equalities of coordinates as $W$ does not satisfy any additional equality of coordinates.

Example 3.2. Consider the following subspaces of $\mathbb{C}^5$:

$$E = \{3x_1 + 4x_2 - x_3 + 3x_5 = 0, x_2 - x_3 - x_4 + x_5 = 0\},$$
$$U = \{x_2 = x_3, x_4 = x_5 = 0, x_1 + x_2 = 0\},$$
$$W = \{x_2 = x_3 = x_4 = x_5, x_1 + 2x_2 = 0\}.$$

Notice that $U$ and $W$ are both 1-dimensional subspaces of $E$ and that

$$P(U) = \{x_2 = x_3, x_4 = x_5\} \quad \text{and} \quad P(W) = \{x_2 = x_3 = x_4 = x_5\}.$$

$U$ is not special in $E$ because $P(W) \subsetneq P(U)$. $W$ is special in $E$ because it satisfies three independent equalities of coordinates, and there is only one 1-dimensional subspace satisfying four independent equalities of coordinates, namely, the fully synchronous subspace, which clearly is not contained in $E$.

After the introduction of this definition it is necessary to take extra care with the word special. The terms special subspace and special Jordan subspace must not be confused. In fact, the first concept is defined for all subspaces of $\mathbb{C}^n$; the second concept is defined only for Jordan subspaces in the context of regular networks. In general, these concepts are distinct. In fact, we can say the following:

1. For regular networks with diagonalizable adjacency matrices, all Jordan subspaces are 1-dimensional.
2. If $G$ is an eigenspace (generalized eigenspace of order 1) that is not associated with the valency of the network, special Jordan subspaces of the network in $G$ are precisely 1-dimensional special subspaces in $G$. Indeed, all Jordan subspaces in $G$ are 1-dimensional, and so a 1-dimensional subspace $J$ of $G$ is not a special Jordan subspace if and only if there is a Jordan subspace $J'$ in $G$ such that $P(J') \subsetneq P(J)$ (recall that $G$ is not associated with the valency). Now, as all Jordan subspaces in $G$ coincide with 1-dimensional subspaces of $G$, this is equivalent to saying that there is a 1-dimensional subspace $J'$ of $G$ such that $P(J') \subsetneq P(J)$, that is, that $J$ is not a special subspace in $G$. Therefore, $J$ is a special Jordan subspace if and only if $J$ is a 1-dimensional special subspace in $G$. 
(3) If $G$ is the eigenspace associated with the valency of the network (recall that the valency is always a semisimple eigenvalue), then
(a) the fully synchronous subspace $F$ is the unique 1-dimensional special subspace in $G$,
(b) all special Jordan subspaces of the network in $G$ are 1-dimensional,
(c) if $\dim G = 2$, then all 1-dimensional subspaces of $G$ are special Jordan subspaces.
If $\dim G > 2$, this fact does not hold (see, for instance, the 4-cell network in Example 4.3).

(4) For generalized eigenspaces $G$ of order greater than 1, not all special subspaces are Jordan subspaces. In fact, Jordan subspaces must be spanned by vectors of Jordan chains. For example, a 1-dimensional special subspace in $G$ is a Jordan subspace if and only if it is a subspace of the corresponding eigenspace (see, for instance, Remark 4.7). In particular, not all special subspaces are special Jordan subspaces.

3.2. Algebraic properties. In this subsection we present important results about special subspaces, which allow us to understand that these subspaces have important algebraic properties.

Proposition 3.3. Consider a $k$-dimensional subspace $E$ of $\mathbb{C}^n$, with $k > 1$. A nonzero subspace $W$ of $E$ is special in $E$ if and only if $W = E \cap P(W)$.

Proof. The result is trivial when $W = E$, and so, henceforward, we suppose that $W$ is a nonzero proper subspace of $E$. We assume first that $W = E \cap P(W)$, and we prove that $W$ is special in $E$. For every subspace $U$ of $E$ satisfying

$$\dim U = \dim W, \quad P(U) \subset P(W),$$

we have

$$U \subset E \cap P(U) \subset E \cap P(W) = W.$$

So, $U = W$ and $P(U) = P(W)$, concluding that $W$ is special in $E$.

Reciprocally, we assume that $W$ is special in $E$, and we prove that $W = E \cap P(W)$. By contradiction, suppose that $W \neq E \cap P(W)$. Then, there is a $(\mu + 1)$-dimensional subspace $\tilde{W}$ of $E \cap P(W)$, where $\mu = \dim W$. Because $\dim \tilde{W} > 1$, there is at least a codimension-1 polydiagonal $X$ that does not contain $\tilde{W}$. So,

$$\tilde{W} \cap X \not\subseteq \tilde{W} \quad \text{and} \quad \operatorname{codim} (\tilde{W} \cap X) = \operatorname{codim} \tilde{W} + 1.$$

Notice the following:
(1) $\dim (\tilde{W} \cap X) = n - (\operatorname{codim} \tilde{W} + 1) = \dim \tilde{W} - 1 = \dim W$.
(2) $P(\tilde{W} \cap X) \subset P(\tilde{W}) \cap X \not\subseteq P(W) \subset P(W)$ (recall that $\tilde{W} \subset P(W)$).
However, these two conditions contradict the fact that $W$ is special in $E$. Therefore, such a subspace $\tilde{W}$ does not exist, and so $W = E \cap P(W)$.

Example 3.4. In Example 3.2, $U$ is not special in $E$ because $U \neq E \cap P(U)$, and $W$ is special in $E$ because trivially $W = E \cap P(W)$.

Next we provide a characterization of special subspaces that will be very useful in the calculation of these subspaces.

Proposition 3.5. Consider a $k$-dimensional subspace $E$ of $\mathbb{C}^n$, with $k > 1$. A nonzero
subspace $W$ of $E$ is special in $E$ if and only if $W = E \cap X$ for some codimension-$\nu$ polydiagonal $X$, with $\nu = \dim E - \dim W$.

**Proof.** The result is trivial when $W = E$, and so, henceforward, we suppose that $W$ is a nonzero proper subspace of $E$. We assume first that $W = E \cap X$, for some polydiagonal $X$, and we prove that $W$ is special in $E$. Under this assumption, we have $P(W) \subset X$, and so

$$W \subset E \cap P(W) \subset E \cap X = W.$$  

Thus, $W = E \cap P(W)$, and, by Proposition 3.3, $W$ is special in $E$.

Conversely, we assume that $W$ is special in $E$. Due to Proposition 3.3, $W = E \cap P(W)$. The equalities of $P(W)$ in the subspace $E$ define a homogeneous linear system with $s$ equations, $s = \operatorname{codim} P(W)$. The rank of this homogeneous system equals $r$, where $r = \operatorname{codim} (E \cap P(W)) - \operatorname{codim} E$, and thus, among all $s$ equalities of $P(W)$, we can choose just $r$ independent equalities to obtain $E \cap X$. Hence, there is a polydiagonal $X$ such that $E \cap X = E \cap P(W) = W$ and

$$\operatorname{codim} X = r = \operatorname{codim} (E \cap P(W)) - \operatorname{codim} E.$$  

This implies that $\operatorname{codim} X = \dim E - \dim W$, concluding the proof.

**Example 3.6.** In Example 2.3, $G_{-1}$ has exactly three 1-dimensional special subspaces, namely, $W_1$, $W_2$, and $W_3$. They are the subspaces that are obtained by intersecting $G_{-1}$ with all codimension-1 polydiagonals defined by a unique equality of coordinates independent from the equalities defining $G_{-1}$. Notice that it suffices to consider three polydiagonals, for example, $\{x_1 = x_2\}$, $\{x_1 = x_4\}$, and $\{x_2 = x_4\}$.

The following result is crucial to proving Theorem 4.9, which is the main result of this work about special Jordan subspaces.

**Proposition 3.7.** If a nonzero subspace $E$ of $\mathbb{C}^n$ does not contain the fully synchronous subspace, then there is a direct sum decomposition of $E$ into 1-dimensional special subspaces in $E$.

**Proof.** Let $k = \dim E$. The result is straightforward when $k = 1$, and so, in what follows we assume $k > 1$. The $n - 1$ independent equalities of the fully synchronous subspace $F$ in the subspace $E$ define a homogeneous linear system with $n - 1$ equations. The rank of this homogeneous system equals $r$, where $r = \operatorname{codim} (E \cap F) - \operatorname{codim} E = n - \operatorname{codim} E = \dim E = k$. Thus, we can choose $k$ equalities of coordinates to obtain $E \cap F = \{0\}$. So, there is a polydiagonal $X$ satisfying

$$E \cap X = E \cap F = \{0\}, \quad \operatorname{codim} X = k.$$  

Let $X_1, \ldots, X_k$ be the $k$ possible distinct codimension-$(k-1)$ polydiagonals that contain $X$. Note that for every $1 \leq i \leq k$ the intersection $E \cap X_i$ is a 1-dimensional special subspace in $E$ and that

$$i \neq j \quad \Rightarrow \quad E \cap X_i \cap X_j = E \cap X = \{0\}.$$  

Therefore, $E = (E \cap X_1) \oplus \cdots \oplus (E \cap X_k)$ is a direct sum decomposition of $E$ into 1-dimensional special subspaces in $E$.

**Corollary 3.8.** Consider a subspace $E$ of $\mathbb{C}^n$ and a nonzero special subspace $W$ in $E$. If $E$ does not contain the fully synchronous subspace, then there is a direct sum decomposition of $W$ into 1-dimensional special subspaces in $E$. 

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Proof. Let \( k = \dim W \). The result is straightforward when \( k = 1 \), and so in what follows we assume \( k > 1 \). Proposition 3.7 guarantees the existence of a direct sum decomposition of \( W \) into 1-dimensional special subspaces in \( W \):

\[
W = W_1 \oplus \cdots \oplus W_k.
\]

Consider \( 1 \leq i \leq k \). Since \( W_i \) is special in \( W \) and \( W \) is special in \( E \), Proposition 3.3 implies that

\[
W_i = P(W_i) \cap W \quad \text{and} \quad W = P(W) \cap E.
\]

Consequently, \( W_i = P(W_i) \cap P(W) \cap E \). Finally, \( P(W_i) \subseteq P(W) \) because \( W_i \subseteq W \), and so \( W_i = P(W_i) \cap E \), allowing us to conclude that \( W_i \) is special in \( E \). \( \blacksquare \)

4. Study of special Jordan subspaces. In this section we consider regular networks, and we study special Jordan subspaces of these networks, considering separately the following situations:

(1) eigenspaces that are not associated with the valency,

(2) eigenspace associated with the valency,

(3) generalized eigenspaces of order greater than 1.

4.1. Special Jordan subspaces in eigenspaces that are not associated with the valency. Recall that if \( G \) is an eigenspace (generalized eigenspace of order 1) that is not associated with the valency of the network, then a subspace \( J \) of \( G \) is a 1-dimensional special subspace in \( G \) if and only if it is a special Jordan subspace of the network. So, the method for listing all special Jordan subspaces in \( G \) consists solely of calculating all 1-dimensional special subspaces in \( G \) (see Examples 2.3 and 3.6).

4.2. Special Jordan subspaces in the eigenspace associated with the valency. Given a regular network, consider the eigenspace \( G \) associated with the valency. If \( G \) is 1-dimensional, then \( G \) is the fully synchronous subspace. If \( G \) has a higher dimension, there are an infinite number of special Jordan subspaces in \( G \). Nevertheless, we prove that the set of the smallest polydiagonals containing these special Jordan subspaces is finite.

Proposition 4.1. Given a regular network, let \( G \) be the eigenspace associated with the valency, with \( \dim G > 1 \), and let \( E \) be a direct complement to the fully synchronous subspace \( F \) in \( G \). Then the following hold:

(1) For every special Jordan subspace \( J \neq F \) in \( G \) there is a special subspace \( J' \) in \( E \) such that \( P(J) = P(J') \) and \( F \oplus J = F \oplus J' \).

(2) A subspace of \( E \) is a special Jordan subspace of the network if and only if it is a special subspace in \( E \).

(3) If \( S = F \oplus J_1 \oplus \cdots \oplus J_k \) is a direct sum of 1-dimensional special Jordan subspaces in \( G \), then there are special subspaces \( J'_1, \ldots, J'_k \) in \( E \) such that \( S = F \oplus J'_1 \oplus \cdots \oplus J'_k \).

Proof. (1) Let \( J \) be a special Jordan subspace of the network in \( G \), \( J \neq F \). Notice that \( J \) is 1-dimensional. If \( J \) is a subspace of \( E \), then it is clear from Definition 2.2 that \( J \) is special in \( E \). So, assume that \( J \) is not a subspace of \( E \). Then,

\[
J = \text{span}\{w\} \quad \text{for some } w \in G, w \notin E, w \notin F.
\]
Because $G = F \oplus E$, we have $w = f + u$ for some $f \in F$ and $u \in E \setminus \{0\}$. Thus,

$$J \subset F \oplus \text{span}\{u\} \quad \text{and} \quad P(J) = P(\text{span}\{u\}).$$

Therefore, $J' = \text{span}\{u\}$ is also a special Jordan subspace of the network in $E$, and so it is a special subspace in $E$. Moreover,

$$J \subset F \oplus J' \Rightarrow F \oplus J \subset F \oplus J',$$

and, because $\dim(F \oplus J) = \dim(F \oplus J')$, we conclude that $F \oplus J = F \oplus J'$.

(2) It is straightforward that every special Jordan subspace in $E$ is a special subspace in $E$. Reciprocally, let $W$ be a special subspace in $E$ and $J$ be a special Jordan subspace in $G$ such that $P(J) \subset P(W)$. Due to (1), there is a special subspace $J'$ in $E$ such that $P(J) = P(J')$, and so $P(J') \subset P(W)$. Therefore, since $W$ is special in $E$, $P(W) = P(J')$ and $W$ is a special Jordan subspace of the network.

(3) If $S = F \oplus J_1 \oplus \cdots \oplus J_k$ is a direct sum of 1-dimensional special Jordan subspaces in $G$, then, due to (1), there are special subspaces $J'_1, \ldots, J'_k$ in $E$ such that $F \oplus J_i = F \oplus J'_i$ for all $1 \leq i \leq k$. Thus, $S = F + J'_1 + \cdots + J'_k$. Besides, $\dim(F + J'_1 + \cdots + J'_k) = \dim S = k + 1$ and so $S = F \oplus J'_1 \oplus \cdots \oplus J'_k$.

**Remark 4.2.** This result shows that, despite the infinite number of special Jordan subspaces in $G$ when $\dim G > 1$, the set of the smallest polydiagonals containing these subspaces is finite, and it consists of the fully synchronous subspace $F$ and the smallest polydiagonals containing all special subspaces in an arbitrary direct complement $E$ to $F$ in $G$. Moreover, a direct sum of $F$ with other special Jordan subspaces in $G$ can be reduced to a direct sum of $F$ with special subspaces in $E$. So, in the context of direct sums of special Jordan subspaces in $G$ involving $F$, we can assume that there is a finite number of special Jordan subspaces in $G$, namely, $F$ and the special subspaces in a fixed direct complement to $F$ in $G$.

**Example 4.3.** Consider the two regular networks given in Figure 4.

![Figure 4. Regular networks for which the valency is a multiple eigenvalue.](image)

For the 3-cell network, the valency is an eigenvalue with multiplicity 2 (algebraic and geometric), and the corresponding eigenspace is

$$G_2 = \{x_1 - 2x_2 + x_3 = 0\}.$$  

As remarked above, all 1-dimensional subspaces in this eigenspace are special Jordan subspaces of the network. According to Proposition 4.1, for every 1-dimensional subspace $J_1$ of $G_2$ distinct from the fully synchronous subspace, $P(J_1)$ is the total phase space.
For the 4-cell network, the valency is an eigenvalue with multiplicity 3, and the corresponding eigenspace is

\[ G_3 = \{ x_1 - 3x_2 + x_3 + x_4 = 0 \}. \]

There are an infinite number of special Jordan subspaces of this network in \( G_3 \). However, using Proposition 4.1 and choosing the direct complement

\[ E = \{ x_1 + x_3 + x_4 = 0, x_2 = 0 \}, \]

we have that for every (1-dimensional) special Jordan subspace \( J_2 \) in \( G_3 \), \( P(J_2) = P(J') \), where \( J' \) is one of the following special subspaces in \( E \):

\[ \{ x_1 = x_2 = 0, x_3 + x_4 = 0 \}, \quad \{ x_1 = x_3, x_2 = 0, 2x_1 + x_4 = 0 \}, \]
\[ \{ x_2 = x_3 = 0, x_1 + x_4 = 0 \}, \quad \{ x_1 = x_4, x_2 = 0, 2x_1 + x_3 = 0 \}, \]
\[ \{ x_2 = x_4 = 0, x_1 + x_3 = 0 \}, \quad \{ x_3 = x_4, x_2 = 0, x_1 + 2x_3 = 0 \}. \]

Proposition 4.1 is useful in proving the following result, which is valid for general eigenspaces (generalized eigenspaces of order 1).

**Proposition 4.4.** Given a regular network, consider an eigenspace \( G \) of the corresponding adjacency matrix. If \( W \) is a nonzero special subspace in \( G \), then there is a direct sum decomposition of \( W \) into special Jordan subspaces of the network.

**Proof.** If \( G \) is not associated with the valency of the network, then, by Corollary 3.8, there is a direct sum decomposition of \( W \) into 1-dimensional special subspaces in \( G \), which are special Jordan subspaces of the network (in \( W \)). If \( G \) is associated with the valency of the network and \( W \) is a special subspace in \( G \), then we first prove that the fully synchronous subspace \( F \) is contained in \( W \). In fact, \( W = P(W) \cap G \), by Proposition 3.3, and \( P(W + F) = P(W) \) because all vectors in \( F \) have equal coordinates. Therefore,

\[ W \subset W + F \subset P(W + F) \cap G = P(W) \cap G = W, \]

and so \( F \subset W \). Finally, we prove that in this case there is a direct sum decomposition of \( W \) into special Jordan subspaces. If \( W = F \), the result is obvious because \( F \) is a special Jordan subspace. Otherwise, we can consider a direct complement \( E \) to \( F \) in \( W \), that is, a nonzero subspace \( E \) of \( W \) such that \( W = F \oplus E \) and, by Corollary 3.8, a direct sum decomposition of \( E \) into 1-dimensional special subspaces in \( W \):

\[ E = E_1 \oplus \cdots \oplus E_s \quad \text{for some } s \geq 1. \]

By Proposition 4.1, \( E_1, \ldots, E_s \) are special Jordan subspaces, and \( W = F \oplus E_1 \oplus \cdots \oplus E_s \) is a direct sum decomposition of \( W \) into (1-dimensional) special Jordan subspaces of the network. \( \blacksquare \)

**Remark 4.5.** In particular, when \( W = E \), this result allows us to conclude that, for a given regular network, there is a direct sum decomposition of each eigenspace (generalized eigenspace of order 1) of the corresponding adjacency matrix into special Jordan subspaces of the network. Later, in Theorem 4.9, we extend this result for generalized eigenspaces of order greater than 1.
4.3. Special Jordan subspaces in generalized eigenspaces of order greater than 1. Given a regular network, consider a generalized eigenspace $G_\lambda$ of order greater than 1. In this subsection we guarantee the existence of special Jordan subspaces that contain specific 1-dimensional Jordan subspaces.

The method of obtaining all $k$-dimensional special Jordan subspaces in $G_\lambda$ can use, for example, the fact that every subspace is contained in a $k$-dimensional special subspace in $\ker(A - \lambda I)^k$, where $A$ is the adjacency matrix of the network. Furthermore, every subspace must contain a $(k-1)$-dimensional Jordan subspace, and thus the calculations can be simplified by demanding the corresponding inclusion between the smallest polydiagonals containing each of these two subspaces.

$$ A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} $$

**Figure 5.** Regular network (left) and the corresponding adjacency matrix (right).

**Example 4.6.** Consider the 5-cell regular network in Figure 5 and the generalized eigenspaces of the corresponding adjacency matrix:

- $G_2 = \{x_1 = x_2 = x_3 = x_4 = x_5\}$,
- $G_1 = \{x_2 = x_5, x_3 = x_4, x_1 = 0, x_2 + x_3 = 0\}$,
- $G_{-1} = \{3x_1 + 4x_2 - x_3 + 3x_5 = 0, x_2 - x_3 - x_4 + x_5 = 0\}$.

$G_2$ and $G_1$ are 1-dimensional, and therefore they are special Jordan subspaces of the network. $G_{-1}$ is a generalized eigenspace of order 2 and

$$ K^1 = \ker(A + I) = \{x_2 = x_3, x_4 = x_5, x_1 + x_2 + x_4 = 0\}. $$

All 1-dimensional special Jordan subspaces in $G_{-1}$ are precisely 1-dimensional special subspaces in $K^1$, and they were already obtained in Example 2.3:

1. $W_{-1,1} = \{x_1 = x_2 = x_3, x_4 = x_5, 2x_1 + x_4 = 0\}$,
2. $W_{-1,2} = \{x_1 = x_4 = x_5, x_2 = x_3, 2x_1 + x_2 = 0\}$,
3. $W_{-1,3} = \{x_2 = x_3 = x_4 = x_5, x_1 + 2x_2 = 0\}$.

Since $G_{-1}$ is a (3-dimensional) generalized eigenspace of order 2, there are 2-dimensional special Jordan subspaces. To obtain these subspaces, notice that

$$ K^1 \cap \im(A + I) = \{x_2 = x_3, x_4 = x_5, x_1 + x_2 + x_4 = 0\} \cap \{x_2 = x_5, x_3 = x_4\} = W_{-1,3}. $$

Thus, all 2-dimensional special Jordan subspaces contain $W_{-1,3}$. Therefore, if $J_2$ is a 2-dimensional special Jordan subspace in $G_{-1}$, then $P(W_{-1,3}) \subset P(J_2)$. There are exactly four
2-dimensional special subspaces in $G_{-1}$ containing $W_{-1,3}$:

\[
K^1 = \{x_2 = x_3, x_4 = x_5, x_1 + x_2 + x_4 = 0\},
\]

\[
J^2_{-1,1} = \{x_2 = x_4, x_3 = x_5, 3x_1 + 4x_2 + 2x_3 = 0\},
\]

\[
J^2_{-1,2} = \{x_2 = x_5, 3x_1 + 7x_2 - x_3 = 0, 2x_2 - x_3 - x_4 = 0\},
\]

\[
J^2_{-1,3} = \{x_3 = x_4, 3x_1 + 4x_2 - x_3 + 3x_5 = 0, x_2 - 2x_3 + x_5 = 0\}.
\]

The first subspace is the 2-dimensional eigenspace of $G_{-1}$, and so it is not a Jordan subspace. The last three subspaces are indeed 2-dimensional Jordan subspaces because they are invariant. Besides, no 2-dimensional Jordan subspace satisfies either $x_2 = x_3$ or $x_4 = x_5$ because

\[
G_{-1} \cap \{x_2 = x_3\} = G_{-1} \cap \{x_4 = x_5\} = K^1.
\]

Thus, we conclude that there are exactly eight special Jordan subspaces of this network:

\[
G_2, \ G_1, \ W_{-1,1}, \ W_{-1,2}, \ W_{-1,3}, \ J^2_{-1,1}, \ J^2_{-1,2}, \ J^2_{-1,3}.
\]

**Remark 4.7.** Notice that in generalized eigenspaces $G$ of order greater than 1, not all special subspaces in $G$ are special Jordan subspaces of the network. For instance, in the previous example, \(\{x_1 = x_2 = x_4, x_3 = x_5, 7x_1 + 2x_3 = 0\}\) is a 1-dimensional special subspace in $G_{-1}$ that is not a Jordan subspace (because it is not spanned by an eigenvector), and thus it is not a special Jordan subspace in $G_{-1}$.

The next result guarantees the existence of special Jordan subspaces that contain specific 1-dimensional Jordan subspaces.

**Theorem 4.8.** Given an $n$-cell regular network with adjacency matrix $A$, consider a generalized eigenspace $G_\lambda$ of order $r \geq 1$. For a positive integer $j$ set

\[
N^j = \text{Ker}(A - \lambda I) \cap \text{Im}(A - \lambda I)^{j-1},
\]

considering $\text{Im}(A - \lambda I)^0 = \mathbb{C}^n$. For every $1 \leq k \leq r$ and for every 1-dimensional special subspace in $N^k$ there is a $k$-dimensional special Jordan subspace containing it.

**Proof.** The result is trivial when $r = 1$, and thus we assume $r > 1$. Consider a positive integer $k \leq r$ and a 1-dimensional special subspace $J_1$ in $N^k$. By Proposition 3.5, consider a polydiagonal $Q$ satisfying

\[
J_1 = Q \cap N^k, \quad P(J_1) \subset Q, \quad \text{codim } Q = \text{dim } N^k - 1.
\]

Consider the subspace

\[
J'_2 = Q \cap (A - \lambda I)^{-1}(J_1) \cap \text{Im}(A - \lambda I)^{k-2}.
\]

All equalities defining $Q$ are independent from all equalities defining $N^k$, and so they are also independent from all equalities defining any subspace containing $N^k$. Hence the following hold:

1. \[
\text{codim } J'_2 = \text{codim } Q + \text{codim } [(A - \lambda I)^{-1}(J_1) \cap \text{Im}(A - \lambda I)^{k-2}]
\]
   \[= \text{dim } N^k - 1 + n - (\text{dim } N^{k-1} + 1)
\]
   \[= n - (\text{dim } N^{k-1} - \text{dim } N^k + 2),
\]

2. \[
\text{codim } (J'_2 \cap K^1) = \text{codim } (J'_2 \cap N^{k-1})
\]
   \[= n - (\text{dim } N^{k-1} - \text{dim } N^k + 1).
\]

Thus,
Applying the same procedure, we obtain a 4-dimensional Jordan subspace $J_1$. This Jordan subspace $J_2$ is contained in $\text{Im} (A - \lambda I)^{k-2}$, and thus we can consider the subspace $J_3 = Q_j \cap (A - \lambda I)^{-1}(J_2) \cap \text{Im} (A - \lambda I)^{k-3}$.

Analogously, there is a system $B_2$ of linearly independent vectors of $N^{k-2}$ such that $N^{k-2} = \text{span}(B_2) \oplus N^k$ and $J_3 = \text{span}(B_2) \oplus J_2 \oplus \text{span}\{x_3\}$, with

$$x_3 \in (A - \lambda I)^{-1}(J_2) \cap \text{Im} (A - \lambda I)^{k-3} \cap (K^2 \setminus K^1).$$

So, $J_3 = J_2 \oplus \text{span}\{x_3\}$ is a Jordan subspace containing $J_1$ and satisfying $P(J_3) \cap N^k = J_1$.

Applying the same procedure, we obtain a 4-dimensional Jordan subspace $J_4$ containing $J_1$ and satisfying $P(J_4) \cap N^k = J_1$. Continuing this process successively, we obtain a $k$-dimensional Jordan subspace $J_k$ containing $J_1$ and satisfying $P(J_k) \cap N^k = J_1$.

It remains to prove that there is a special Jordan subspace containing $J_1$. Let $Y_k$ be a $k$-dimensional special Jordan subspace of the network such that $P(Y_k) \subset P(J_k)$. Then,

$$Y_k \cap N^k \subset P(Y_k) \cap N^k \subset P(J_k) \cap N^k = J_1,$$

and so the 1-dimensional subspace $Y_k \cap N^k$ is $J_1$. Therefore, $Y_k$ is a special Jordan subspace containing $J_1$.

This theorem guarantees that in a generalized eigenspace of order $r \geq 1$ each 1-dimensional special subspace in $N^k$ is contained in a $k$-dimensional special Jordan subspace, with $1 \leq k \leq r$. Since, for each such $k$, there is a 1-dimensional special subspace in $N^k$, it follows that this theorem also guarantees the existence of special Jordan subspaces of all possible dimensions: 1 up to $r$.

### 4.4. Main theorem on special Jordan subspaces

In this subsection we prove the main result of this work on special Jordan subspaces.

**Theorem 4.9.** Given an $n$-cell regular network, there is a direct sum decomposition of $\mathbb{C}^n$ into special Jordan subspaces of the network.

**Proof.** $\mathbb{C}^n$ is the direct sum of all generalized eigenspaces [4, p. 47], and thus it suffices to prove that every generalized eigenspace $G_\lambda$ admits a direct sum decomposition into special Jordan subspaces of the network.

If $G_\lambda$ is an eigenspace, the result follows by Proposition 4.4.

If $G_\lambda$ is a generalized eigenspace of order $r > 1$, let

$$N^i = \text{Ker} (A - \lambda I) \cap \text{Im} (A - \lambda I)^{i-1} \text{ and } \nu_i = \dim N^i,$$

with $1 \leq i \leq r$. First, by Proposition 3.7, consider a direct sum decomposition of $N^r$ into 1-dimensional special subspaces of $N^r$. Since $N^r \subset N^{r-1}$, we can add $(\nu_{r-1} - \nu_r)$ 1-dimensional
special subspaces in \(N^{r-1}\) to the previous sum and obtain a direct sum decomposition of \(N^{r-1}\). Continuing this process, we obtain a direct sum decomposition of \(N^{r-1}\), that includes, for each \(1 \leq i \leq r\), exactly \((\nu_i - \nu_{i+1})\) 1-dimensional special subspaces of \(N^i\), considering \(\nu_{r+1} = 0\). Finally, by Theorem 4.8, for each special subspace \(W_j\) in \(N^i \setminus N^{i-1}\), \(1 \leq i \leq r\), we can consider an \(i\)-dimensional special Jordan subspace \(J_j\) containing \(W_j\), considering \(N^0 = \{0\}\). The set of the constructed special Jordan subspaces is linearly independent [11, p. 233], and so \(G_{\lambda} = J_1 \oplus \cdots \oplus J_{\nu_1}\) is a direct sum decomposition of the whole generalized eigenspace \(G_{\lambda}\).

**Remark 4.10.** If the adjacency matrix of the network has at least one generalized eigenspace of order greater than 1, the special Jordan subspaces in this direct sum decomposition are not necessarily 1-dimensional. For instance, in Example 4.6,

\[
\mathbb{C}^5 = G_2 \oplus G_1 \oplus W_{-1,1} \oplus J_{2,-1,1}^2
\]

is a direct sum decomposition of \(\mathbb{C}^5\) into special Jordan subspaces of the 5-cell network in Figure 5. Furthermore, notice that, using Theorem 2.2.4 of Gohberg, Lancaster, and Rodman [4], Theorem 4.9 can be stated in a longer version, as follows.

*Given an \(n\)-cell regular network, there exists a direct sum decomposition

\[
\mathbb{C}^n = J_1 \oplus \cdots \oplus J_p,
\]

where \(J_i\) is a special Jordan subspace of the network corresponding to an eigenvalue \(\lambda_i\) of the adjacency matrix (here \(\lambda_1, \ldots, \lambda_p\) are not necessarily different). Moreover, if \(\mathbb{C}^n = J'_1 \oplus \cdots \oplus J'_q\) is another direct sum decomposition with special Jordan subspaces \(J'_i\) corresponding to eigenvalues \(\mu_i\), \(i = 1, \ldots, q\), then \(q = p\), and (possibly after a permutation of \(J'_1, \ldots, J'_q\)) \(\dim J_i = \dim J'_i\) and \(\lambda_i = \mu_i\) for \(i = 1, \ldots, q\).

**4.5. Importance of special subspaces in the study of special Jordan subspaces.** The study of special Jordan subspaces presented in this section allows one to understand the importance of special subspaces in this work. In fact, we have seen that, given a regular network, there is the following immediate relationship between these subspaces:

1. In eigenspaces that are not associated with the valency, special Jordan subspaces coincide with 1-dimensional special subspaces.
2. In the eigenspace \(G\) associated with the valency, two different situations may occur: if \(G\) is 1-dimensional, then \(G\) is the fully synchronous subspace \(F\); otherwise, there are an infinite number of special Jordan subspaces in \(G\). Nevertheless, only a finite number is necessary to obtain all synchrony subspaces (in the context of direct sums involving \(F\), as explained in Remark 4.2). To be precise, this finite list of special Jordan subspaces in \(G\) consists of \(F\) together with all 1-dimensional special subspaces in a fixed arbitrary direct complement to \(F\) in \(G\).
3. In generalized eigenspaces \(G'\) of order greater than 1, there are special Jordan subspaces with dimension greater than 1. In the general case, the relation between special Jordan subspaces and special subspaces is direct only in the 1-dimensional case, since 1-dimensional special Jordan subspaces in \(G'\) coincide with 1-dimensional special subspaces in the eigenspace contained in \(G'\). Notice that in higher dimensions, for every special Jordan subspace \(J\) in \(G'\), there is a \(k\)-dimensional special subspace \(W\) in \(G'\).
such that \( P(W) \subset P(J) \). (As a matter of fact, in all examples that we have studied, all \( k \)-dimensional special Jordan subspaces are \( k \)-dimensional special subspaces, but we do not know whether this curious fact is always true.)

Special subspaces are also important in this work because their algebraic properties were essential to proving the main result of this work, that is, Theorem 5.1. Notice that, due to the above relation, these algebraic properties are also properties of special Jordan subspaces in many cases.

Finally, special subspaces are also important because they simplify the calculation of special Jordan subspaces. In fact, given a regular network, it is easy to identify, directly by definition, all special Jordan subspaces just by looking at the list of all Jordan subspaces. However, the calculation of the complete list of all Jordan subspaces can be a tedious task. The simplest method consists of using Proposition 3.5 to obtain all special subspaces, and the above relation to obtain the list of all special Jordan subspaces.

5. List of all synchrony subspaces. In this section we relate special Jordan subspaces with synchrony.

5.1. Main theorem on synchrony. In this subsection we prove the main result of this work on synchrony.

**Theorem 5.1.** Given a regular network, a polydiagonal is a synchrony subspace if and only if it is a direct sum of special Jordan subspaces of the network.

**Proof.** By Lemma 1.1, it suffices to prove that every synchrony subspace admits a direct sum decomposition into special Jordan subspaces. Let \( G \) be an \( n \)-cell regular network, \( S \) a synchrony subspace, \( Q \) the corresponding \( m \)-cell quotient network, and let \( A \) and \( A_Q \) be the adjacency matrices of \( G \) and \( Q \), respectively. By Theorem 4.9, there is a direct sum decomposition of \( C^m, C^m = J_1 \oplus \cdots \oplus J_m \), into special Jordan subspaces of \( Q \).

Consider the isomorphism \( \phi \) from \( C^m \) into \( S \) defined by the natural identification between these two spaces, and notice that for every real \( \mu \)

\[
A - \mu I_n = \phi \circ (A_Q - \mu I_m) \circ \phi^{-1}.
\]

\( S \) can be written as

\[
S = \phi(C^m) = \phi(J_1 \oplus \cdots \oplus J_s) = \phi(J_1) \oplus \cdots \oplus \phi(J_m).
\]

Notice that if \( i \neq j \), \( \phi(J_i) \cap \phi(J_j) = \{0\} \). Next we prove that the last equality represents a direct sum decomposition of \( S \) into special Jordan subspaces. For each \( i = 1, \ldots, m \) we have the following:

1. \( \phi(J_i) \) is a Jordan subspace of \( A \): assuming that \( \{x_1, \ldots, x_k\} \) is a Jordan chain spanning \( J_i \) in a generalized eigenspace \( G_{\lambda_i} \), then, for \( 1 \leq j \leq k \),

\[
\phi(x_j) \neq 0, \quad (A - \lambda_i I_n)(\phi(x_j)) = \phi((A_Q - \lambda_i I_m)(x_j)) = \phi(x_{j-1}),
\]

considering \( x_0 = 0 \). So, \( \{\phi(x_1), \ldots, \phi(x_k)\} \) is a Jordan chain in \( S \) spanning \( \phi(J_i) \).

2. \( \phi(J_i) \) is a special Jordan subspace of \( G \): let \( \bar{X}_i \) be a special Jordan subspace of \( G \) such that \( P(\bar{X}_i) \subset P(\phi(J_i)) \). \( \bar{X}_i \) is contained in \( S \) because \( \bar{X}_i \subset P(\bar{X}_i) \subset P(\phi(J_i)) \subset S \),
and so we can consider the pre-image \( X_i = \phi^{-1}(\tilde{X}_i) \). Analogously to what was done in (1), we prove that \( X_i \) is a Jordan subspace of \( A_Q \). Therefore,

\[
P(\tilde{X}_i) \subset P(\phi(J_i)) \Rightarrow P(X_i) \subset P(J_i) \Rightarrow X_i = J_i \Rightarrow \tilde{X}_i = \phi(J_i),
\]

and thus \( \phi(J_i) \) is a special Jordan subspace of \( G \).

Hence, \( S = \phi(J_1) \oplus \cdots \oplus \phi(J_m) \) is a direct decomposition of \( S \) into special Jordan subspaces of \( G \). □

Recall that the special Jordan subspaces in this direct sum decomposition are not necessarily 1-dimensional (Remark 4.10).

**Corollary 5.2.** Given a regular network, there are 2-dimensional synchrony subspaces if and only if there are eigenvectors of the corresponding adjacency matrix with exactly two distinct coordinates.

**Proof.** A 2-dimensional synchrony subspace is precisely a direct sum of the fully synchronous subspace with a 1-dimensional special Jordan subspace satisfying \( n - 2 \) independent equalities of coordinates, where \( n \) is the number of cells of the network. □

**Corollary 5.3.** Given an \( n \)-cell regular network, a polydiagonal defined by \( n - l \) equalities of coordinates is a synchrony subspace if and only if these equalities are satisfied by special Jordan subspaces whose sum is \( l \)-dimensional.

**Proof.** First, assume that \( S \) is a synchrony subspace defined by \( n - l \) equalities. Using Theorem 5.1, \( S \) is the direct sum of special Jordan subspaces whose sum is \( l \)-dimensional. Hence, because each of them is contained in \( S \), each of them also satisfies the \( n - l \) equalities of \( S \).

Conversely, assume that there are special Jordan subspaces satisfying \( n - l \) equalities of coordinates and whose sum is \( l \)-dimensional. Then, their sum has codimension \( n - l \) and is defined by the \( n - l \) common equalities of coordinates. Thus, it is polydiagonal. Further, it also invariant because it is the sum of invariant subspaces. So, the sum is a synchrony subspace. □

### 5.2. Method for listing all synchrony subspaces.

Theorem 5.1 and Corollary 5.3 provide a useful method for listing all synchrony subspaces of an \( n \)-cell regular network:

1. Compute all special Jordan subspaces (all possible dimensions). If the eigenspace associated with the valency has dimension greater than 1, consider a finite number of special Jordan subspaces in this eigenspace (Remark 4.2).
2. For each \( 1 \leq l \leq n - 1 \), analyze whether there are special Jordan subspaces with \( n - l \) common equalities of coordinates and whose sum is \( l \)-dimensional. Only in the affirmative case is this sum a synchrony subspace.

**Remark 5.4.** Note that step 2 can be easily done just by looking at the list of all special Jordan subspaces. Then, this method just requires calculations in the first step to obtain that list. For instance, using the list of special Jordan subspaces of the 5-cell network in Figure 5, which was obtained in Example 4.6, we can see that \( x_2 = x_4 \) is an equality of coordinates satisfied by three special Jordan subspaces, namely \( G_2 \), \( G_1 \), and \( J_{2,1,2} \), whose sum is 4-dimensional. Therefore, \( \{x_2 = x_4\} \) is a synchrony subspace of that network.

**Example 5.5.** In Example 2.3, we calculate all special Jordan subspaces of the 5-cell regular network in Figure 3, namely,
For each $1 \leq l \leq 4$, we analyze whether there are special Jordan subspaces with $5 - l$ common equalities of coordinates and whose sum is $l$-dimensional, obtaining exactly five nontrivial synchrony subspaces:

1. $G_2 \oplus W_{-1,1} = \{x_1 = x_2 = x_3, x_4 = x_5\}$,
2. $G_2 \oplus W_{-1,2} = \{x_1 = x_4 = x_5, x_2 = x_3\}$,
3. $G_2 \oplus W_{-1,3} = \{x_2 = x_3 = x_4 = x_5\}$,
4. $G_2 \oplus W_{-1,1} \oplus W_{-1,2} = \{x_2 = x_3, x_4 = x_5\}$,
5. $G_2 \oplus G_i \oplus G_{-i} \oplus W_{-1,2} = \{x_1 = x_4\}$.

The corresponding lattice is presented in Figure 6(left).

**Figure 6.** The lattice of all synchrony subspaces of the networks in Figures 3 (left) and 5 (right). Each cycle $(i_1, \ldots, i_s)$ denotes the equality of the corresponding cell coordinates, and $P$ denotes the total phase space.

**Example 5.6.** In Example 4.6 we listed all special Jordan subspaces of the 5-cell regular network in Figure 5. Applying the method to calculate synchrony subspaces described above, we obtain the synchrony subspaces that appear in the corresponding lattice in Figure 6(right).

### 6. Special Jordan subspaces and lattices of synchrony subspaces

In this section we consider lattices of synchrony subspaces of regular networks, and we show that special Jordan subspaces play an important role in the structure of these lattices.

#### 6.1. The lattice of all synchrony subspaces

Given a linear transformation $A$ from $\mathbb{C}^n$ into $\mathbb{C}^n$, the set of all $A$-invariant subspaces is a lattice, where the meet and the join operations are the intersection and sum, respectively, of sets [4, p. 31]. Stewart proved that the set of all synchrony subspaces of a general network is a complete lattice and that this lattice is not a sublattice of the lattice of all $A$-invariant subspaces, where $A$ is the adjacency matrix of the network [14]. In fact, although the meet operation is the same for both lattices, the join is not because the sum of two synchrony subspaces is not always a synchrony subspace. In fact, it is straightforward that the join of a finite number of elements is the smallest synchrony subspace (that is, intersection of all synchrony subspaces) containing all these elements and,
in particular, containing their sum. It is also straightforward that the sum of synchrony subspaces is a synchrony subspace if and only if the sum is polydiagonal.

**Example 6.1.** Taking this last statement into account, we can understand that the lattice structure $L_{14}$ in Figure 7, presented by Kamei in [8], for 4-cell regular networks whose adjacency matrices have only simple eigenvalues, can be eliminated. This statement answers the query in Remark 6.1 of [8].

![Figure 7. Lattice structure $L_{14}$ of Kamei [8].](image)

In fact, in that work, the $l$th level of a lattice includes all codimension-$(l - 1)$ synchrony subspaces (and so, the descriptions of these lattices are upside-down compared with those of all the other lattices in this paper). Hence, as we see in Figure 7, structure $L_{14}$ is relative to a network with exactly three codimension-2 synchrony subspaces, and so they are exactly three of the following polydiagonals:

- $S_1 = \{x_1 = x_2 = x_3\}$
- $S_2 = \{x_1 = x_2 = x_4\}$
- $S_3 = \{x_1 = x_3 = x_4\}$
- $S_4 = \{x_2 = x_3 = x_4\}$

The figure shows that the sum of two of these is a codimension-1 polydiagonal subspace. Therefore, at least one of them is $S_i$ for some $1 \leq i \leq 4$. But the sum of this synchrony subspace with the third codimension-2 synchrony subspace is also a codimension-1 polydiagonal, a fact that contradicts $L_{14}$. Hence, this structure can be eliminated.

**6.2. Special Jordan subspaces and the lattice structure.** Consider a regular network and the corresponding lattice $L$ of all synchrony subspaces. Theorem 5.1 implies that all special Jordan subspaces are sufficient to list all synchrony subspaces and thus to construct $L$. In subsection 5.2 we present a simple method for obtaining that list. In this subsection we present two other methods for obtaining this lattice, which show that special Jordan subspaces play an important role in the structure of these lattices. Recall that we always assume that there is a finite list of special Jordan subspaces (Remark 4.2).

Both methods use the well-known concept of join-irreducibility from lattice theory. Recall that an element of a lattice is *join-irreducible* if it is neither the bottom element nor the join of two smaller elements (see, for example, [7]). These elements are important because in finite lattices every element distinct from the bottom is a join of join-irreducible elements, and so these elements can generate the whole lattice, from the bottom to the top, using the join operation.

In the first method we show that the lattice $L$ can be obtained from another lattice $L'$. **First method:** Consider the lattice $L'$ for which the join-irreducible elements are precisely the special Jordan subspaces of the network, the bottom element is the zero subspace, and
the join operation is the sum of subspaces. This first method consists solely in selecting the polydiagonals of $L'$ as the elements of $L$. In fact, on the one hand, every synchrony subspace is polydiagonal and, by Theorem 5.1, is a (direct) sum of special Jordan subspaces. Thus, it is an element of $L'$. On the other hand, every polydiagonal of $L'$ is a sum of special Jordan subspaces and so is a sum of invariant subspaces. Hence, every polydiagonal of $L'$ is invariant and, as a consequence, is a synchrony subspace. Therefore, the elements of $L$ are precisely the polydiagonals of $L'$. Notice that $L$ is not a sublattice of $L'$ because the join operations are distinct.

**Remark 6.2.** In some cases, the elements of $L$ are precisely the polydiagonals of a sublattice $L''$ of $L'$, whose set of join-irreducible elements is contained properly in the set of special Jordan subspaces (see the final part of Example A.3).

The second method uses the following result.

**Theorem 6.3.** Given a regular network, consider the lattice of all synchrony subspaces. Every join-irreducible element of this lattice is the smallest synchrony subspace containing at least one special Jordan subspace. Moreover, the number of join-irreducible elements of this lattice does not exceed the number of special Jordan subspaces.

**Proof.** Let $S$ be a join-irreducible element of the lattice. By Theorem 5.1, $S$ can be written as a direct sum of special Jordan subspaces, say $J_1, \ldots, J_k$. Let $S_1, \ldots, S_k$ be the smallest synchrony subspaces containing $J_1, \ldots, J_k$, respectively. Then, $S_1, \ldots, S_k$ are elements of the considered lattice and $S = S_1 + \cdots + S_k$. Since $S$ is a join-irreducible element, $S_i = S$ for some $1 \leq i \leq k$. Thus, $S$ is the smallest synchrony subspace containing at least one special Jordan subspace. As an immediate consequence, we get the last statement of this theorem.

**Example 6.4.** In Example 2.3 there are six special Jordan subspaces and four join-irreducible elements. In Example 4.6 there are eight special Jordan subspaces and seven join-irreducible elements (see the lattices in Figure 6).

**Remark 6.5.** The second part of this result is obvious when the eigenspace associated with the valency of the network has dimension greater than 1 because in this case there are an infinite number of special Jordan subspaces. However, due to Proposition 4.1, this result remains valid even when we consider, for the purpose of finding synchrony subspaces, a finite number of special Jordan subspaces (Remark 4.2).

Theorem 6.3 guarantees that the set of special Jordan subspaces is sufficient to determine all join-irreducible elements of the lattice $L$, and so we can describe another method for constructing this lattice.

**Second method:** Let $J_1, \ldots, J_k$ be the special Jordan subspaces of a given regular network and $S_1, \ldots, S_k$ be the smallest synchrony subspaces containing $J_1, \ldots, J_k$, respectively. For all $1 \leq i \leq k$, a synchrony subspace $S_i$ distinct from the fully synchronous subspace is a join-irreducible element of the lattice $L$ if and only if all proper synchrony subspaces of $S_i$ in $X = \{S_1, \ldots, S_k\}$ are contained in a unique proper synchrony subspace of $S_i$, that is, if and only if

$$\exists l \in \{1, \ldots, k\} \forall j \in \{1, \ldots, k\} \quad (S_j \subsetneq S_i \Rightarrow S_j \subset S_l).$$

(6.1)

Indeed, every synchrony subspace $S$ is the join of a finite number of elements of $X$ and, also, $S \leq S_i \iff S \subset S_i$, where $\leq$ denotes the order of the lattice. This second method consists of using condition (6.1) to identify, among $S_1, \ldots, S_k$, all join-irreducible elements, and then
obtaining all the other elements of \( L \) using the join operation. (Recall that the join of a finite number of elements in \( L \) is the smallest synchrony subspace containing all these elements and, in particular, containing their sum.)

Notice that this second method describes a very easy way of obtaining all join-irreducible elements just by looking at the list of all special Jordan subspaces, without calculations:

- first, we obtain the list of the smallest synchrony subspaces containing at least one special Jordan subspace, using step 2 of the method in subsection 5.2;
- second, we extract from this list only the join-irreducible elements, using condition (6.1).

**Example 6.6.** In Example 2.3 we calculate all special Jordan subspaces of the network in Figure 3. Using step 2 of the method in subsection 5.2, we identify for each special Jordan subspace the corresponding smallest synchrony subspace containing it, and obtain the following list:

\[
\begin{align*}
\{x_1 = x_2 = x_3 = x_4 = x_5\}, \\
\{x_1 = x_2 = x_3, x_4 = x_5\}, \\
\{x_1 = x_4 = x_5, x_2 = x_3\}, \\
\{x_2 = x_3 = x_4 = x_5\}, \\
\{x_1 = x_4\}.
\end{align*}
\]

Using condition (6.1), we conclude that all join-irreducible elements are precisely the nontrivial synchrony subspaces of this list. (See also the final part of Example A.3.)

**Remark 6.7.** As discussed in section 1.2, Aguiar and Dias [1] proved that all synchrony subspaces of a general network can be obtained by using the sum operation over a small set of synchrony subspaces—the sum-irreducible synchrony subspaces—which are the synchrony subspaces that cannot be represented as a sum of proper synchrony subspaces. It is a fact that every join-irreducible element of the lattice of synchrony subspaces is a sum-irreducible synchrony subspace. Indeed, a synchrony subspace that is not sum-irreducible can be represented as a sum of proper synchrony subspaces and is, therefore, the join of these smaller elements. Hence, it is not a join-irreducible element. However, not every sum-irreducible synchrony subspace is a join-irreducible element. Consider, for example, the 5-cell regular network defined by the following adjacency matrix:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

The eigenspaces of this matrix are:

\[
\begin{align*}
E_2 &= \{x_1 = x_2 = x_3 = x_4 = x_5\}, \\
E_{-2} &= \{x_1 = x_2, x_3 = x_4 = x_5, x_1 + x_3 = 0\}, \\
E_{-1} &= \{x_1 = x_3, x_2 = x_4 = x_5, 12x_1 + x_2 = 0\}, \\
E_1 &= \{x_4 = x_5, 12x_1 + x_2 = 0, x_1 + x_3 = 0\}, \\
E_0 &= \{x_1 = x_2 = 0, x_3 = x_4, x_3 + x_5 = 0\},
\end{align*}
\]

and so there are exactly four nontrivial synchrony subspaces:
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(1) \(E_2 \oplus E_{-2} = \{x_1 = x_2, x_3 = x_4 = x_5\}\),
(2) \(E_2 \oplus E_{-1} = \{x_1 = x_3, x_2 = x_4 = x_5\}\),
(3) \(E_2 \oplus E_{-2} \oplus E_0 = \{x_1 = x_2, x_3 = x_4\}\),
(4) \(E_2 \oplus E_{-2} \oplus E_{-1} \oplus E_1 = \{x_4 = x_5\}\).

The synchrony subspace \(\{x_4 = x_5\}\) is not a join-irreducible element of the corresponding lattice, as is clear in Figure 8. However, it is a sum-irreducible synchrony subspace because it cannot be represented as a sum of proper synchrony subspaces. This 5-cell network was obtained from the 4-cell network \#32 of Kamei [9, p. 3718], for which the total phase space \(P\) is also a sum-irreducible synchrony subspace that is not a join-irreducible element of the corresponding lattice.

![Figure 8](image)

**Figure 8.** Lattice of all synchrony subspaces of the networks mentioned in Remark 6.7. Each cycle \((i_1, \ldots, i_s)\) denotes the equality of the corresponding cell coordinates, and \(P\) denotes the total phase space.

Finally, we point out that Theorem 6.3 is also valid if we substitute “join-irreducible elements” by “sum-irreducible synchrony subspaces,” and the proof is essentially the same (considering the same substitution). In particular, if \(J_1, \ldots, J_k\) are all special Jordan subspaces and \(S_1, \ldots, S_k\) are the smallest synchrony subspaces containing \(J_1, \ldots, J_k\), respectively, then, for all \(1 \leq i \leq k\), \(S_i\) is a sum-irreducible synchrony subspace if and only if it cannot be written as a sum of other proper synchrony subspaces in \(X = \{S_1, \ldots, S_k\}\). After identifying, among \(S_1, \ldots, S_k\), all sum-irreducible synchrony subspaces, it is possible to obtain all the other elements of the lattice using the sum operation, as explained in Aguiar and Dias [1].

### 6.3. Number of synchrony subspaces.

Our study shows that a higher number of special Jordan subspaces implies, in general, a higher number of synchrony subspaces because this fact increases the chance that common equalities of coordinates will appear, and thus increases the possibility that sums of special Jordan subspaces that are polydiagonal will also appear. In particular, a higher dimension of an eigenspace implies, in general, a higher number of synchrony subspaces.

Moreover, among networks with the same number of cells, it is natural to expect a higher number of synchrony subspaces and of possible lattice structures in networks with diagonalizable adjacency matrices. To see this, we consider two \(n\)-cell regular networks \(Q_1\) and \(Q_2\), with adjacency matrices \(A_1\) and \(A_2\), for which a subspace \(G\) of \(\mathbb{C}^n\) is an eigenspace and a
generalized eigenspace of order \( r > 1 \), respectively. First, if

\[ S_1 = J_{2,1} \oplus \cdots \oplus J_{2,s_1} \]

is a direct sum of special Jordan subspaces of \( Q_2 \) in \( G \) (subspaces of \( G \)), with \( s_1 \geq 1 \), we can always define a direct sum \( S'_1 \) of special Jordan subspaces of \( Q_1 \) in \( G \) such that

\[ \dim(S'_1) = \dim(S_1) \text{ and } P(S'_1) \subset P(S_1). \]

In fact, by the definition of special subspace, there is a special subspace \( S \) in \( G \) such that \( \dim S = \dim S_1 \) and \( P(S) \subset P(S_1) \). (If \( S_1 \) is special in \( G \), then \( S = S_1 \).) Proposition 4.4 guarantees that there is a direct sum decomposition of \( S \) into \((1\text{-dimensional})\) special Jordan subspaces of \( Q_1 \) in \( G \) and thus guarantees the existence of the direct sum \( S'_1 \). However, if

\[ S_2 = J_{1,1} \oplus \cdots \oplus J_{1,s_2} \]

is a direct sum of special Jordan subspaces of \( Q_1 \) in \( G \), with \( s_2 \geq 1 \), it is not always possible to define a direct sum \( S'_2 \) of special Jordan subspaces of \( Q_2 \) in \( G \) such that

\[ \dim(S'_2) = \dim(S_2) \text{ and } P(S'_2) \subset P(S_2), \]

because special Jordan subspaces of \( Q_2 \) in \( G \) can have dimension greater than 1 and their existence requires the existence of Jordan chains. Notice that the generalized eigenvectors spanning a special subspace can be arbitrary, but the generalized eigenvectors spanning a (special) Jordan subspace must form a Jordan chain.

7. Conclusions. This paper was mainly motivated by the work of Stewart [14] about lattices of synchrony subspaces, and by the work of Kamei [8] about the relation between synchrony subspaces and classes of eigenvectors of the corresponding adjacency matrix. It was also motivated by our observations, in some examples that we were trying, that on the list of all synchrony subspaces written as direct sums of Jordan subspaces, some Jordan subspaces were essential and others were not.

We then proved the existence of a class of Jordan subspaces whose elements were sufficient to generate all synchrony subspaces by direct sums, and defined the elements in this class as special Jordan subspaces. To be more precise, we showed that all synchrony subspaces can be obtained through a small set of Jordan subspaces, by direct sums. We also emphasize the close relationship between the special Jordan subspaces of a regular network and the corresponding lattice structure of synchrony subspaces.

Appendix. We consider four examples of regular networks, and, for each case, we list all synchrony subspaces and present the corresponding lattice.

Example A.1. Consider the 3-cell regular network in Figure 4 and the eigenspaces of the corresponding adjacency matrix:

\[ G_2 = \{x_1 - 2x_2 + x_3 = 0\}, \quad G_0 = \{x_1 = x_3 = 0\}. \]

Due to Proposition 4.1, to calculate all synchrony subspaces we can assume that the list of all special Jordan subspaces in \( G_2 \) is finite, and without loss of generality we consider that the
fully synchronous subspace $F$ and $\text{span}\{(2, 1, 0)\}$ are the unique special Jordan subspaces in this eigenspace. The eigenspace $G_0$ is 1-dimensional, and thus it is a special Jordan subspace. Therefore, there is only one nontrivial synchrony subspace, namely, $\{x_1 = x_3\} = F \oplus G_0$. The corresponding lattice is presented in Figure 9(left).

**Example A.2.** Consider the 4-cell regular network in Figure 4 and the eigenspaces of the corresponding adjacency matrix:

$$G_3 = \{x_1 - 3x_2 + x_3 + x_4 = 0\}, \quad G_0 = \{x_1 = x_3 = x_4 = 0\}.$$  

Due to Proposition 4.1, we can assume that the list of all special Jordan subspaces in $G_3$ is finite, and, using the calculations of Example 4.3, we can assume that the special Jordan subspaces in $G_3$ are the following:

1. $W_{3,1} = \{x_1 = x_2 = x_3 = x_4\}$,
2. $W_{3,2} = \{x_1 = x_2 = 0, x_3 + x_4 = 0\}$,
3. $W_{3,3} = \{x_1 = x_3, x_2 = 0, 2x_1 + x_4 = 0\}$,
4. $W_{3,4} = \{x_1 = x_4, x_2 = 0, 2x_1 + x_3 = 0\}$,
5. $W_{3,5} = \{x_2 = x_3 = 0, x_1 + x_4 = 0\}$,
6. $W_{3,6} = \{x_2 = x_4 = 0, x_1 + x_3 = 0\}$,
7. $W_{3,7} = \{x_3 = x_4, x_2 = 0, x_1 + 2x_3 = 0\}$.

The eigenspace $G_0$ is 1-dimensional and thus is a special Jordan subspace. Therefore, there are eight special Jordan subspaces of the network, and we obtain exactly four nontrivial synchrony subspaces:

1. $W_{3,1} \oplus G_0 = \{x_1 = x_3 = x_4\}$,
2. $W_{3,1} \oplus W_{3,4} \oplus G_0 = \{x_1 = x_4\}$,
3. $W_{3,1} \oplus W_{3,3} \oplus G_0 = \{x_1 = x_3\}$,
4. $W_{3,1} \oplus W_{3,7} \oplus G_0 = \{x_3 = x_4\}$.

The corresponding lattice is presented in Figure 9(right).

**Example A.3.** Consider the regular network in Figure 10 and the eigenspaces of the corresponding adjacency matrix:

1. $G_2 = \{x_1 = x_2 = x_3 = x_4 = x_5\}$,
There are exactly thirteen special Jordan subspaces of this network:

(1) $G_2 = \{x_1 = x_2 = x_3 = x_4 = x_5\}$,
(2) $G_1 = \{x_1 = x_2 = x_4 = 0, x_3 = x_5\}$,
(3) $G^{-1} = \{x_1 + x_3 + x_5 = 0, x_1 + x_2 + x_4 = 0\}$.

For each $1 \leq l \leq 4$, we analyze whether there are special Jordan subspaces with $5 - l$ common equalities of coordinates and whose sum is $l$-dimensional, obtaining exactly sixteen nontrivial synchrony subspaces:

(1) $G_2 \oplus G_1 = \{x_1 = x_2 = x_4 = x_5\}$,
(2) $G_2 \oplus W_{-1,1} = \{x_1 = x_2 = x_3, x_4 = x_5\}$,
(3) $G_2 \oplus W_{-1,3} = \{x_1 = x_2 = x_3, x_3 = x_4\}$,
(4) $G_2 \oplus W_{-1,4} = \{x_1 = x_3 = x_4, x_2 = x_5\}$,
(5) $G_2 \oplus W_{-1,6} = \{x_1 = x_4 = x_5, x_2 = x_3\}$,
(6) $G_2 \oplus W_{-1,7} = \{x_2 = x_3 = x_4 = x_5\}$,
(7) $G_2 \oplus G_1 \oplus W_{-1,2} = \{x_1 = x_2 = x_4\}$,
(8) $G_2 \oplus G_1 \oplus W_{-1,7} = \{x_2 = x_4, x_3 = x_5\}$,
(9) $G_2 \oplus G_1 \oplus W_{-1,8} = \{x_1 = x_2, x_3 = x_5\}$,
(10) $G_2 \oplus G_1 \oplus W_{-1,10} = \{x_1 = x_4, x_3 = x_5\}$,
(11) $G_2 \oplus W_{-1,1} \oplus W_{-1,6} = \{x_2 = x_3, x_4 = x_5\}$,
(12) $G_2 \oplus W_{-1,3} \oplus W_{-1,4} = \{x_2 = x_5, x_3 = x_4\}$,
(13) $G_2 \oplus G_1 \oplus W_{-1,1} \oplus W_{-1,2} = \{x_1 = x_2\}$,
(14) $G_2 \oplus G_1 \oplus W_{-1,2} \oplus W_{-1,4} = \{x_1 = x_4\}$,
(15) $G_2 \oplus G_1 \oplus W_{-1,2} \oplus W_{-1,7} = \{x_2 = x_4\}$,
(16) $G_2 \oplus G_1 \oplus W_{-1,5} \oplus W_{-1,7} = \{x_3 = x_5\}$.
The corresponding lattice is presented in Figure 11. Notice that there are thirteen special Jordan subspaces and nine join-irreducible elements in this lattice.

**Figure 11.** The lattice of all synchrony subspaces of the network in Figure 10. Each cycle \((i_1, \ldots, i_s)\) denotes the equality of the corresponding cell coordinates, and \(P\) denotes the total phase space.

We use this example to point out two important facts about the methods presented in subsection 6.2:

1. All synchrony subspaces are precisely the polydiagonals of the lattice \(L''\), whose bottom element is the zero subspace; the join operation is the sum of subspaces; and the join-irreducible elements are the special Jordan subspaces excluding \(W_{-1,5}, W_{-1,9},\) and \(W_{-1,11}\).

2. Right after obtaining the list of all special Jordan subspaces, it is possible to identify all join-irreducible elements of the lattice (without calculating all synchrony subspaces and constructing the lattice). First, using step 2 of the method in subsection 5.2, we identify for each special Jordan subspace the corresponding smallest synchrony subspace containing it, and obtain the following list:

   (a) \(S_1 = \{x_1 = x_2 = x_3 = x_4 = x_5\}\),
   (b) \(S_2 = \{x_1 = x_2 = x_4, x_3 = x_5\}\),
   (c) \(S_3 = \{x_1 = x_2 = x_3, x_4 = x_5\}\),
   (d) \(S_4 = \{x_1 = x_2 = x_4\}\),
   (e) \(S_5 = \{x_1 = x_2 = x_5, x_3 = x_4\}\),
   (f) \(S_6 = \{x_1 = x_3 = x_4, x_2 = x_5\}\),
   (g) \(S_7 = \{x_3 = x_5\}\),
   (h) \(S_8 = \{x_1 = x_4 = x_5, x_2 = x_3\}\),
   (i) \(S_9 = \{x_2 = x_3 = x_4 = x_5\}\),
   (j) \(S_{10} = \{x_1 = x_2, x_3 = x_5\}\),
   (k) \(S_{11} = \{x_2 = x_4\}\),
   (l) \(S_{12} = \{x_1 = x_4, x_3 = x_5\}\).

After that, we use condition (6.1) of subsection 6.2 to obtain the join-irreducible elements, concluding that they are precisely all these synchrony subspaces excluding \(S_1, S_7,\) and \(S_{11}\).
Example A.4. Consider the regular network in Figure 12. The corresponding adjacency matrix has two different eigenvalues, namely, 1 and 0, and the corresponding generalized eigenspaces are

\[ G_1 = \{ x_1 = x_2 = x_3 = x_4 = x_5 = x_6 \}, \quad G_0 = \{ x_1 = 0 \}. \]

\[ G_1 \text{ is a 1-dimensional eigenspace, and thus it is a special Jordan subspace.} \]
\[ G_0 \text{ is a generalized eigenspace of order 3, and so there are } k \text{-dimensional special Jordan subspaces of the network in } G_0 \text{ for } 1 \leq k \leq 3. \]

Moreover,
\[ K^1 = \ker(A) = \{ x_1 = x_2 = x_4 = x_5 = 0 \}, \]
\[ K^2 = \ker(A^2) = \{ x_1 = x_4 = 0 \}. \]

All 1-dimensional special Jordan subspaces of the network in \( G_0 \) are precisely all 1-dimensional special subspaces in \( K^1 \), namely,

1. \( W_1 = \{ x_1 = x_2 = x_3 = x_4 = x_5 = 0 \} \)
2. \( W_2 = \{ x_1 = x_2 = x_4 = x_5 = x_6 = 0 \} \)
3. \( W_3 = \{ x_1 = x_2 = x_4 = x_5, x_3 = x_6 = 0 \} \)

To obtain all 2-dimensional special Jordan subspaces in \( G_0 \), notice that
\[ K^1 \cap \text{Im}(A) = K^1 \cap \{ x_1 = x_2 = x_4 \} = K^1. \]

We calculate all 2-dimensional special Jordan subspaces using Theorem 4.8. This theorem guarantees the existence of 2-dimensional special Jordan subspaces containing each one of the 1-dimensional special subspaces in \( N^2 = K^1 \). The possible special Jordan subspaces containing \( W_1 \) are obtained with the calculation of all 2-dimensional special subspaces \( J_2 \) in the pre-image
\[ A^{-1}(W_1) = \{ x_1 = x_2 = x_4 = 0 \}, \]
satisfying \( P(W_1) \subset P(J_2) \) and \( J_2 \cap (K^2 \setminus K^1) \neq \emptyset \), leading to
\[ U_1 = \{ x_1 = x_2 = x_3 = x_4 = 0 \} \supset W_1, \]
\[ U_2 = \{ x_1 = x_2 = x_4 = 0, x_3 = x_5 \} \supset W_1. \]

Applying the same procedure to the other two pre-images,
\[ A^{-1}(W_2) = \{ x_1 = x_4 = x_5 = 0 \}, \]
\[ A^{-1}(W_3) = \{ x_1 = x_4 = 0, x_2 = x_5 \}, \]
we obtain the following three additional Jordan subspaces:

\[ U_3 = \{x_1 = x_4 = x_5 = x_6 = 0\} \supset W_2, \]
\[ U_4 = \{x_1 = x_4 = x_5 = 0, x_2 = x_6\} \supset W_2, \]
\[ U_5 = \{x_1 = x_4 = 0, x_2 = x_5, x_3 = x_6\} \supset W_3. \]

These subspaces are 2-dimensional special subspaces in \(K^2\), and thus they are special Jordan subspaces of the network.

There are no more 2-dimensional special Jordan subspaces. Indeed, if \(J\) is a 2-dimensional Jordan subspace, then \(P(J)\) contains \(P(W_i)\) for some \(1 \leq i \leq 3\). If \(P(W_1) = \{x_1 = x_2 = x_3 = x_4 = x_5\} \subset P(J)\), we must take into account that \(K^2 = \{x_1 = x_4 = 0\}\) and that

\[ U_1 = \{x_1 = x_2 = x_3 = x_4 = 0\}, \quad U_2 = \{x_1 = x_2 = x_4 = 0, x_3 = x_5\} \]

are special Jordan subspaces satisfying \(P(W_1) \subset P(U_j)\) with \(1 \leq j \leq 2\). Thus, we just have to prove that if

\[ P(J) = \{x_1 = x_4 = x_5\} \quad \text{or} \quad P(J) = \{x_1 = x_4, x_2 = x_5\}, \]

then \(J\) is not a special Jordan subspace. But that trivially follows from the existence of \(U_3\) and \(U_5\), respectively. A similar situation occurs when \(P(W_2) \subset P(J)\) and \(P(W_3) \subset P(J)\).

To obtain all 3-dimensional special Jordan subspaces, notice that

\[ K^2 \cap \text{Im}A = \{x_1 = x_4 = 0\} \cap \{x_1 = x_2 = x_4 = 0\} = \{x_1 = x_2 = x_4 = 0\}. \]

Therefore, calculating all 3-dimensional special subspaces \(J_3\) in the pre-images,

\[ A^{-1}(U_1) = \{x_1 = x_2 = 0\} \quad \text{and} \quad A^{-1}(U_2) = \{x_1 = 0, x_2 = x_4\}, \]

satisfying \(P(U_1) \subset P(J_3)\) and \(P(U_2) \subset P(J_3)\), respectively, and \(J_3 \cap (G_0 \setminus K^2) \neq \emptyset\), we obtain the following three 3-dimensional Jordan subspaces:

\[ V_1 = \{x_1 = x_2 = x_3 = 0\}, \]
\[ V_2 = \{x_1 = x_2 = 0, x_3 = x_4\}, \]
\[ V_3 = \{x_2 = x_4, x_3 = x_5, x_1 = 0\}. \]

Analogously, it is proved that these are 3-dimensional special Jordan subspaces and that they are the unique subspaces in this condition.

Hence, we obtain the following list of special Jordan subspaces:

1. \(G_1 = \{x_1 = x_2 = x_3 = x_4 = x_5 = x_6\}\),
2. \(W_1 = \{x_1 = x_2 = x_3 = x_4 = 0\}\),
3. \(W_2 = \{x_1 = x_2 = x_4 = x_5 = 0\}\),
4. \(W_3 = \{x_1 = x_2 = x_4 = x_5, x_3 = x_6 = 0\}\),
5. \(U_1 = \{x_1 = x_2 = x_3 = x_4 = 0\}\),
6. \(U_2 = \{x_1 = x_2 = x_4 = 0, x_3 = x_5\}\),
7. \(U_3 = \{x_1 = x_4 = x_5 = x_6 = 0\}\),
8. \(U_4 = \{x_1 = x_4 = x_5 = 0, x_2 = x_6\}\),
(9) \( U_5 = \{ x_1 = x_4 = 0, x_2 = x_5, x_3 = x_6 \} \),
(10) \( V_1 = \{ x_1 = x_2 = x_3 = 0 \} \),
(11) \( V_2 = \{ x_1 = x_2 = 0, x_3 = x_4 \} \),
(12) \( V_3 = \{ x_2 = x_4, x_3 = x_5, x_1 = 0 \} \).

For each \( 1 \leq l \leq 5 \), we analyze whether there are special Jordan subspaces with \( 6 - l \) common equalities of coordinates and whose sum is \( l \)-dimensional, obtaining exactly 18 nontrivial synchrony subspaces:

(1) \( G_1 \oplus W_1 = \{ x_1 = x_2 = x_3 = x_4 = x_5 \} \),
(2) \( G_1 \oplus W_2 = \{ x_1 = x_2 = x_4 = x_5 = x_6 \} \),
(3) \( G_1 \oplus W_3 = \{ x_1 = x_2 = x_4 = x_5, x_3 = x_6 \} \),
(4) \( G_1 \oplus W_1 \oplus W_2 = \{ x_1 = x_2 = x_4 = x_5 \} \),
(5) \( G_1 \oplus U_1 = \{ x_1 = x_2 = x_3 = x_4 \} \),
(6) \( G_1 \oplus U_2 = \{ x_1 = x_2 = x_4, x_3 = x_5 \} \),
(7) \( G_1 \oplus U_3 = \{ x_1 = x_4 = x_5 = x_6 \} \),
(8) \( G_1 \oplus U_4 = \{ x_1 = x_4 = x_5, x_2 = x_6 \} \),
(9) \( G_1 \oplus U_5 = \{ x_1 = x_4, x_2 = x_5, x_3 = x_6 \} \),
(10) \( G_1 \oplus W_2 \oplus U_1 = \{ x_1 = x_2 = x_4 \} \),
(11) \( G_1 \oplus W_2 \oplus U_3 = \{ x_1 = x_4 = x_5 \} \),
(12) \( G_1 \oplus W_1 \oplus U_5 = \{ x_1 = x_4, x_2 = x_5 \} \),
(13) \( G_1 \oplus V_1 = \{ x_1 = x_2 = x_3 \} \),
(14) \( G_1 \oplus V_2 = \{ x_1 = x_2, x_3 = x_4 \} \),
(15) \( G_1 \oplus V_3 = \{ x_2 = x_4, x_3 = x_5 \} \),
(16) \( G_1 \oplus W_2 \oplus V_1 = \{ x_1 = x_2 \} \),
(17) \( G_1 \oplus U_1 \oplus U_3 = \{ x_1 = x_4 \} \),
(18) \( G_1 \oplus W_2 \oplus V_3 = \{ x_2 = x_4 \} \).

The lattice of all synchrony subspaces is presented in Figure 13. Notice that in this case there are 12 special Jordan subspaces and 11 join-irreducible elements.

![Diagram](image)

**Figure 13.** The lattice of all synchrony subspaces of the network in Figure 12. Each cycle \((i_1, \ldots, i_n)\) denotes the equality of the corresponding cell coordinates, and \( P \) denotes the total phase space.
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